Surminimisation of Automata

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A few starting remarks

- Alphabets are provided with a total order.

In examples: $a < b < c$, $x < y < z$ and $0 < 1 < 2 < 3 < \cdots$

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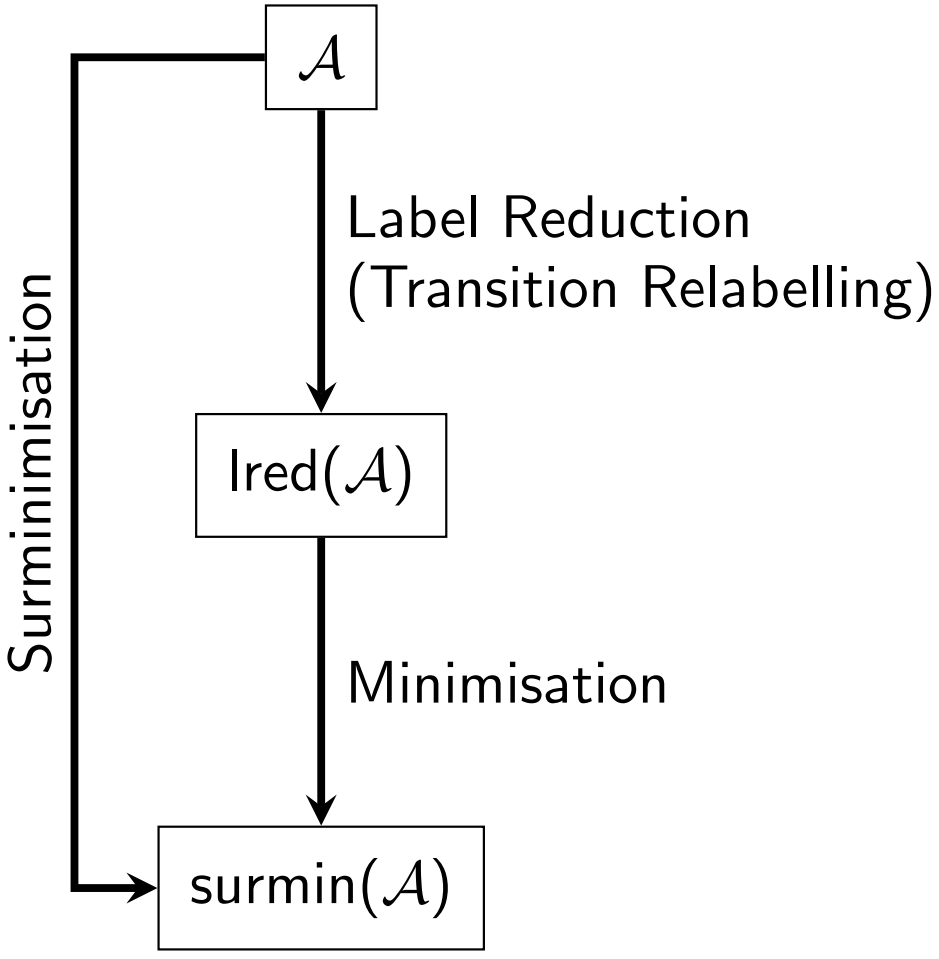
- Automata are deterministic.
- Automata are trim.

In particular, *minimisation*, *minimal* refers to a trim automaton.

Outline

Automaton Surminimisation



Definition (Label Reduction of Automata)

Automaton $\mathcal{A} = \langle \mathsf{Q}, \mathsf{A}, \delta, \mathsf{i}, \mathsf{F} \rangle$

Its label-reduction: $\mathsf{lred}(\mathcal{A}) = \langle \mathsf{Q}, \mathsf{B}, \delta', \mathsf{i}, \mathsf{F} \rangle$

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$$\left.\begin{array}{lcl} p & \xrightarrow{a_0} & p_0 \\ p & \xrightarrow{a_1} & p_1 \\ & \vdots & \\ p & \xrightarrow{a_k} & p_k \end{array}\right\} \text{ with } a_0 < a_1 < \cdots < a_k \ .$$

$\Big\downarrow$ Label Reduction

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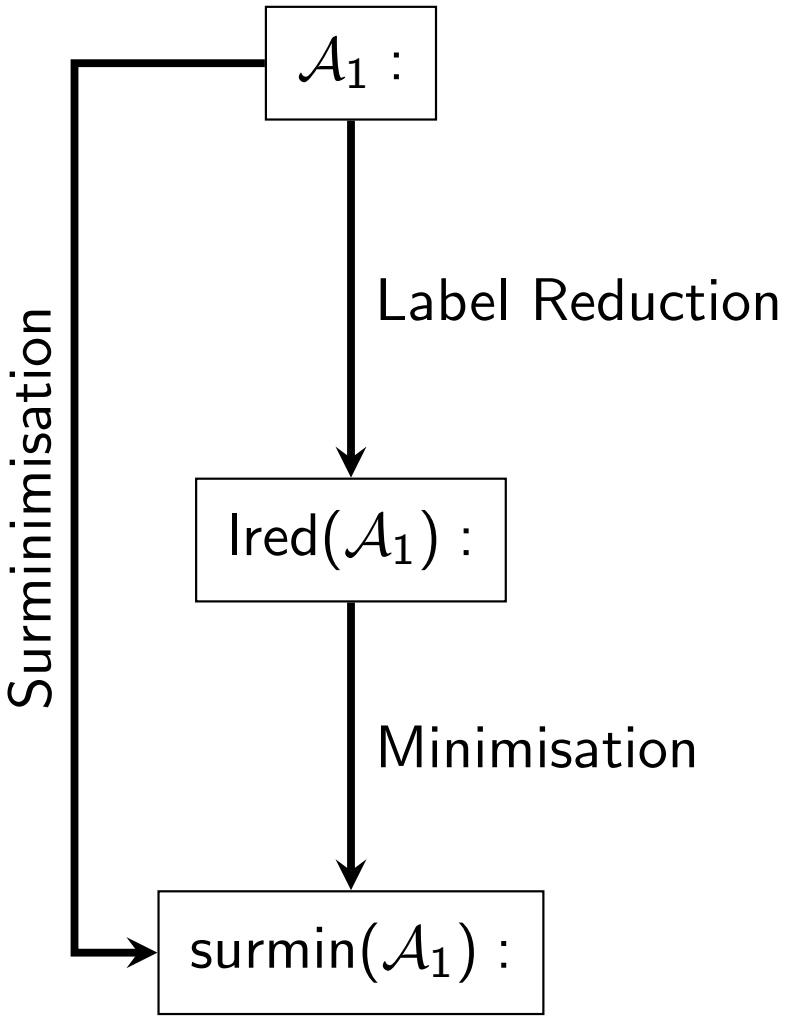
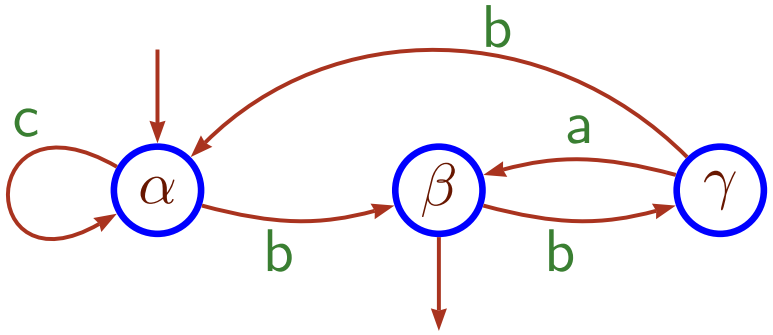
$B = \{0, 1, \cdots (k-1)\}$ where k is the maximal out-degree of the states of $\mathcal{A}$

Example of Surminimisation (1)





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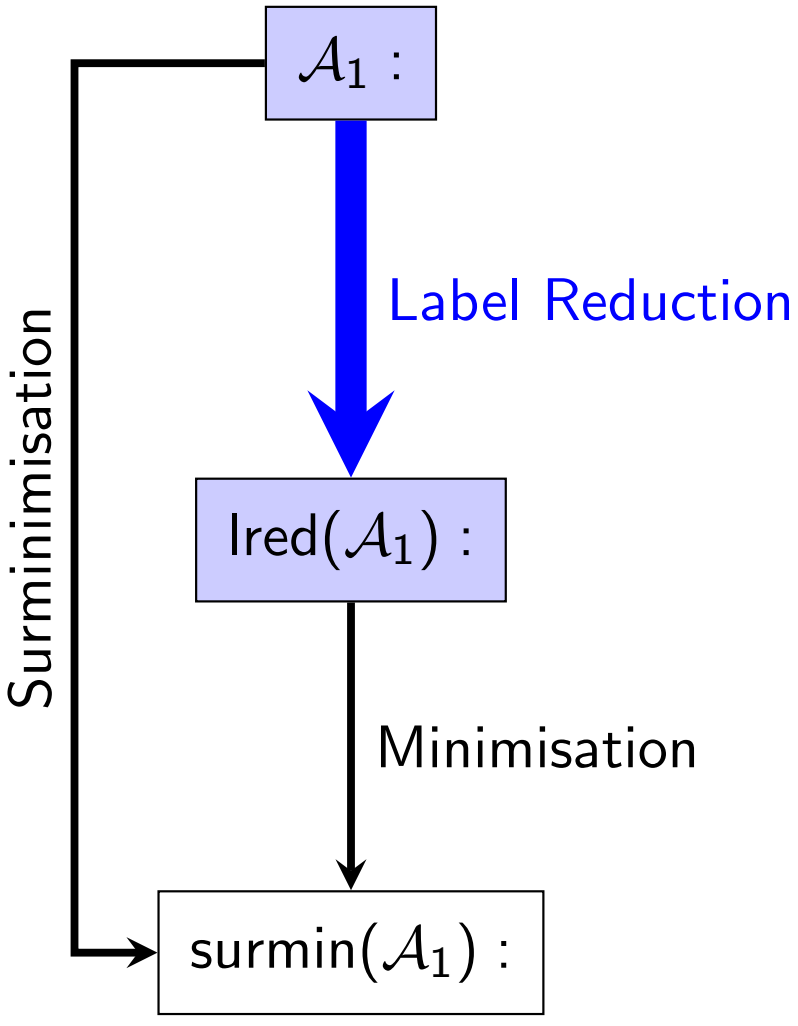

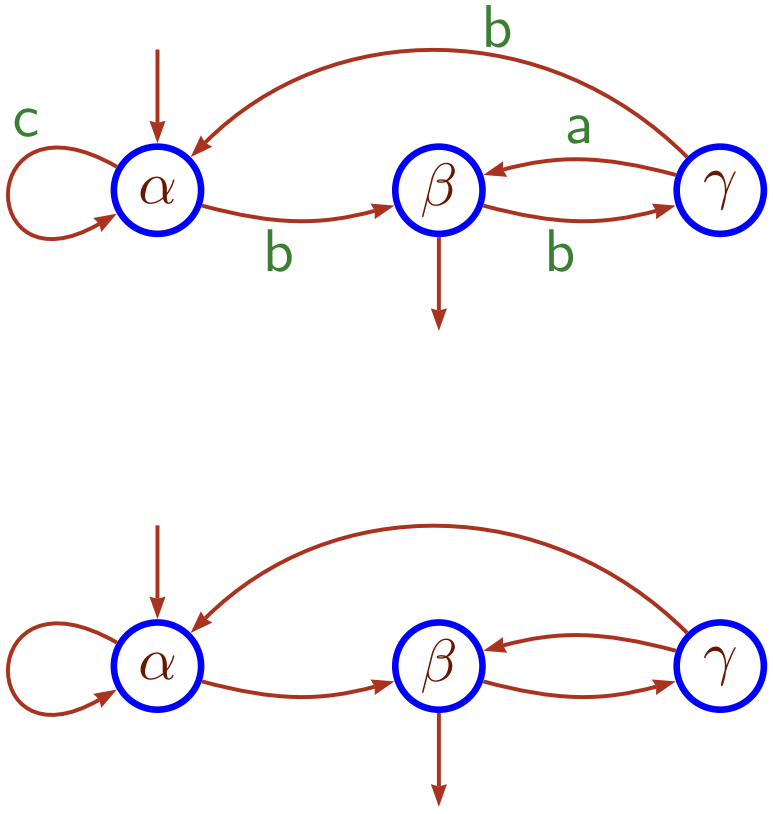

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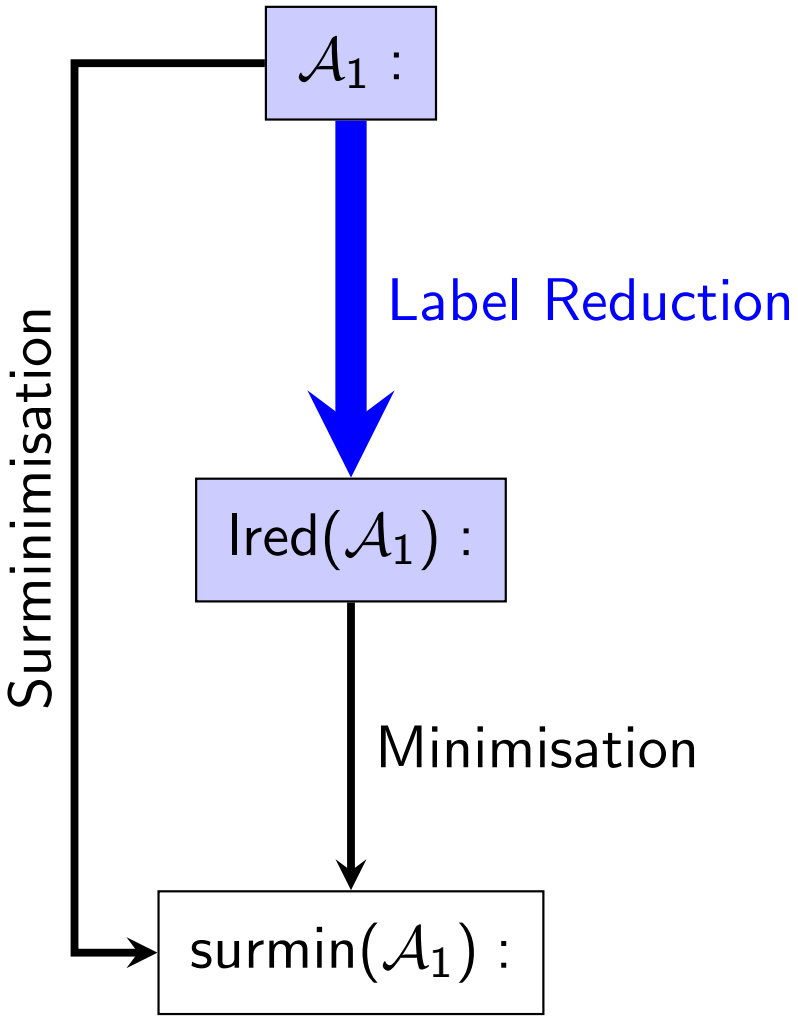

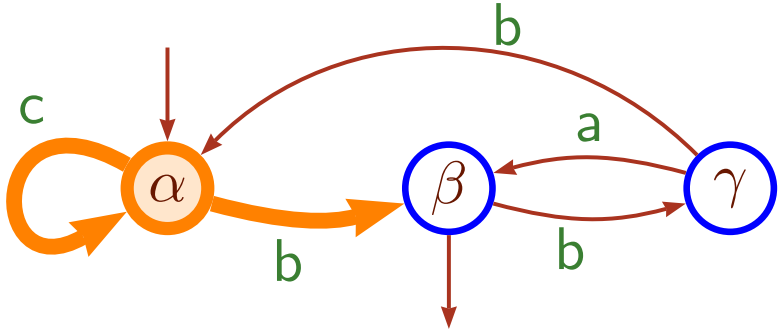

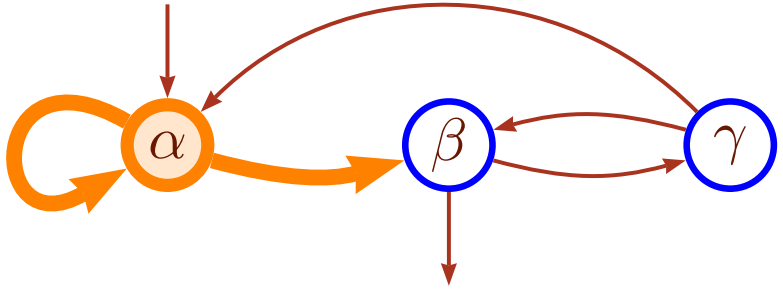

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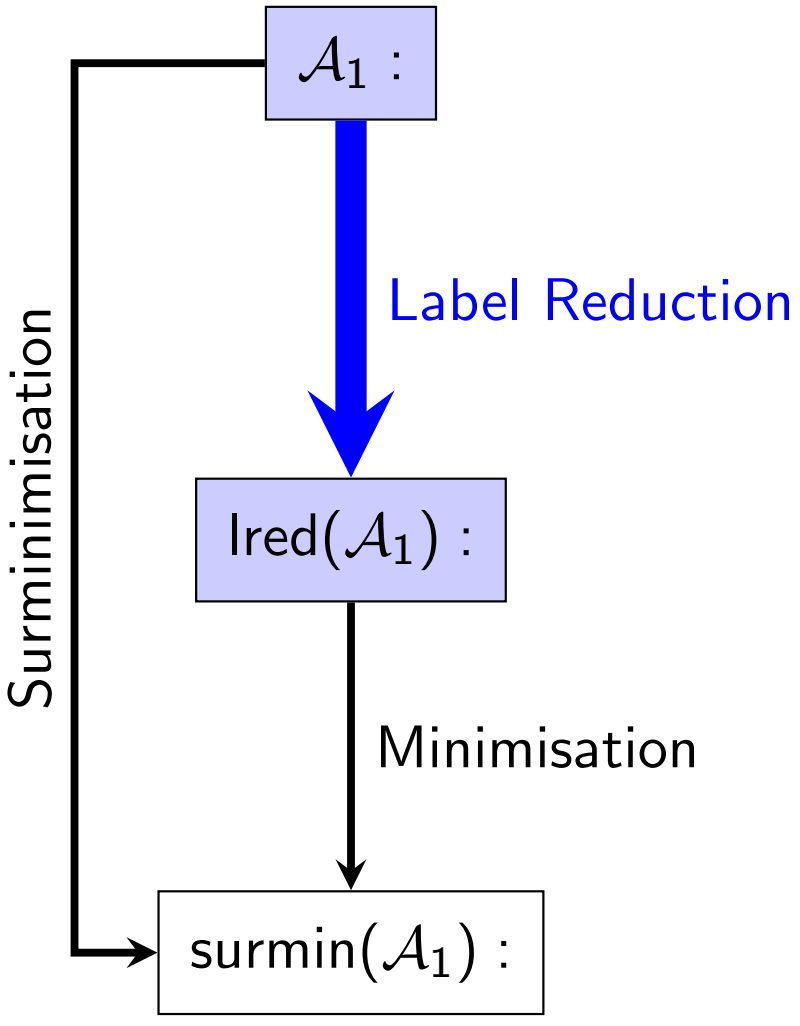

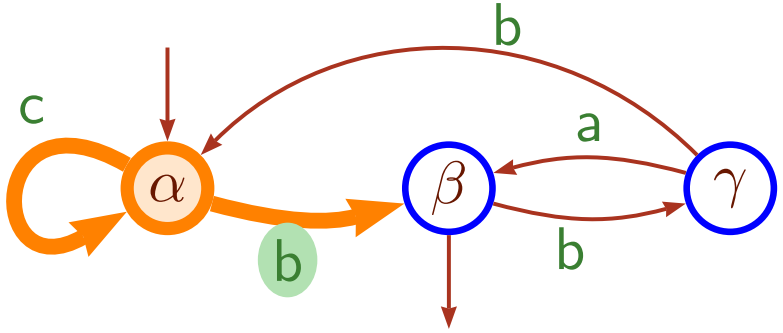

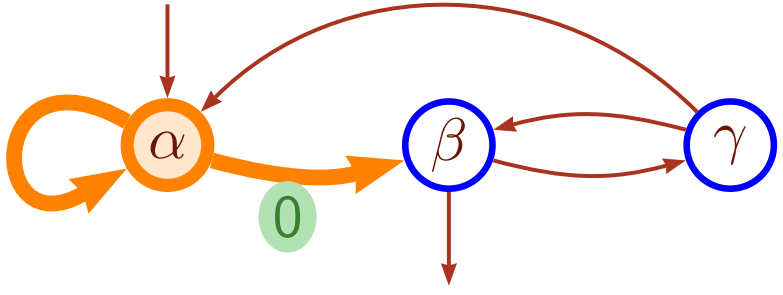

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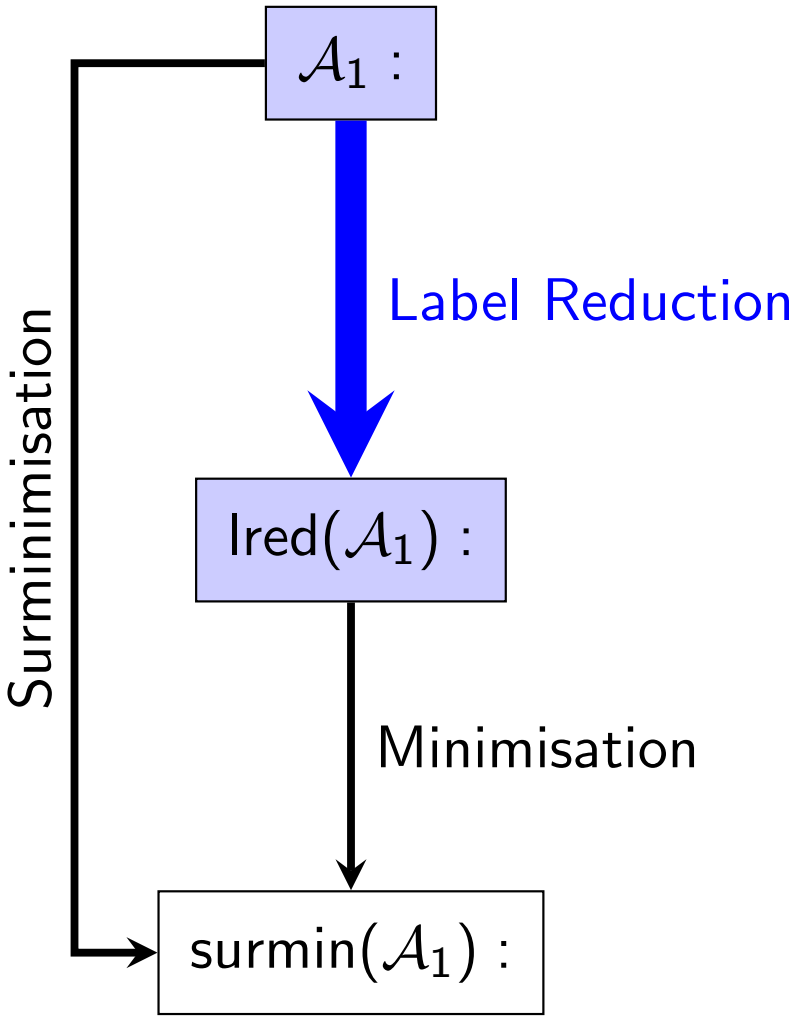

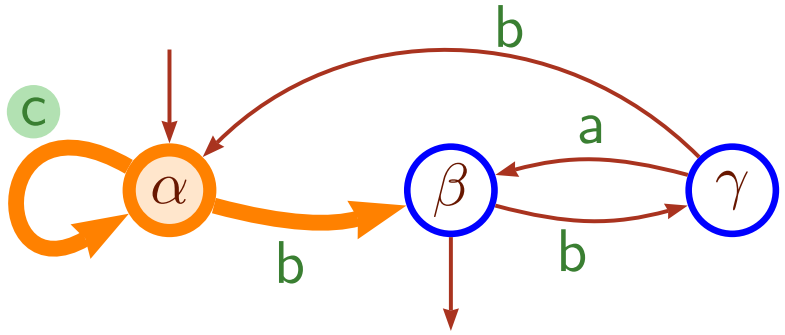

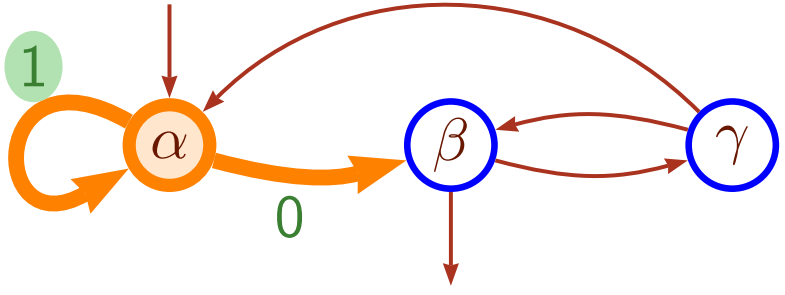

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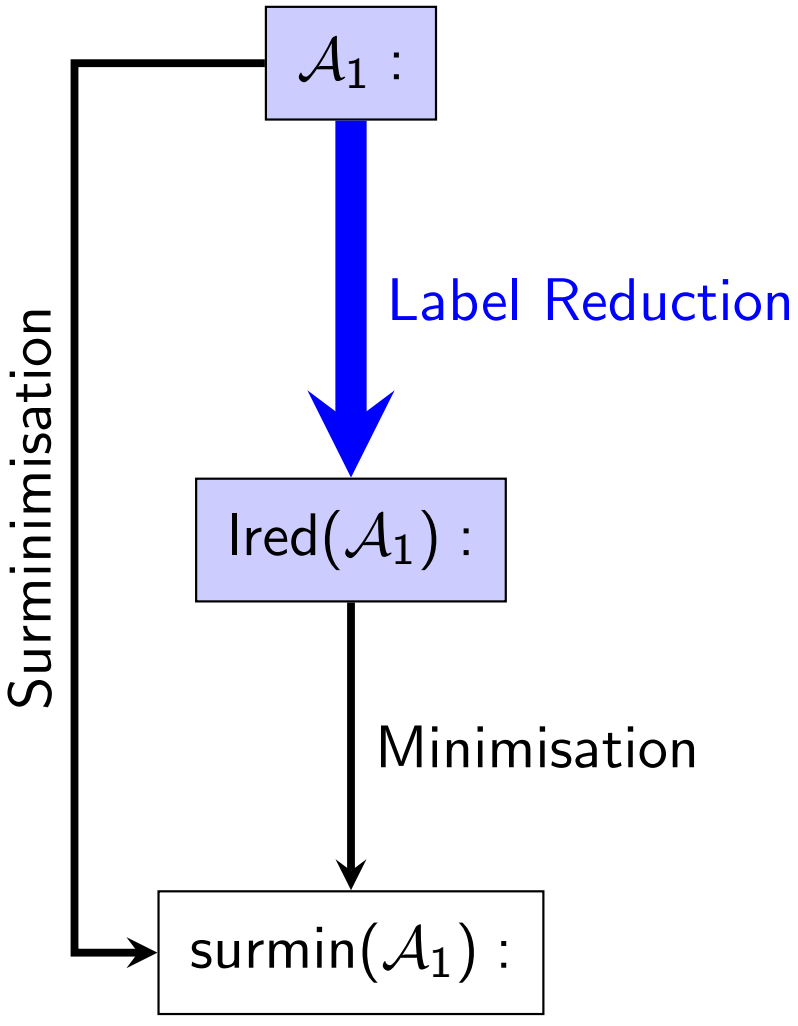

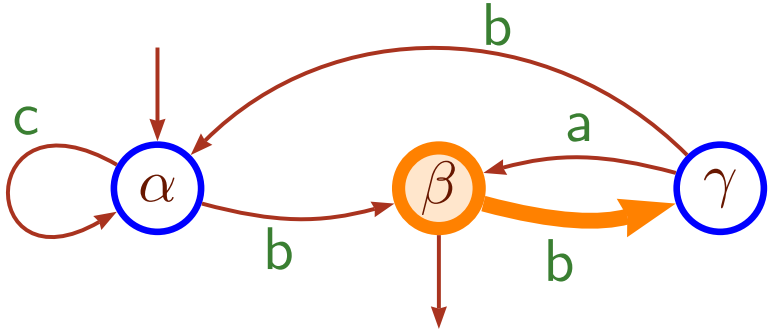

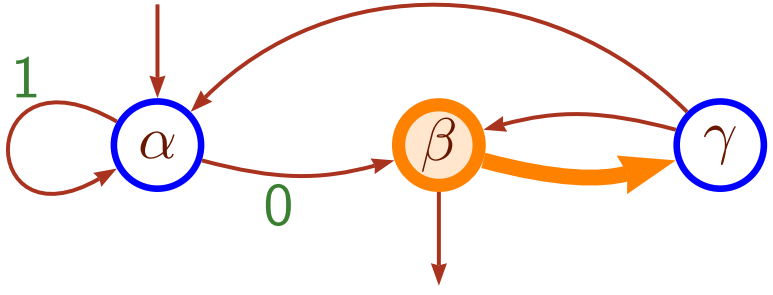

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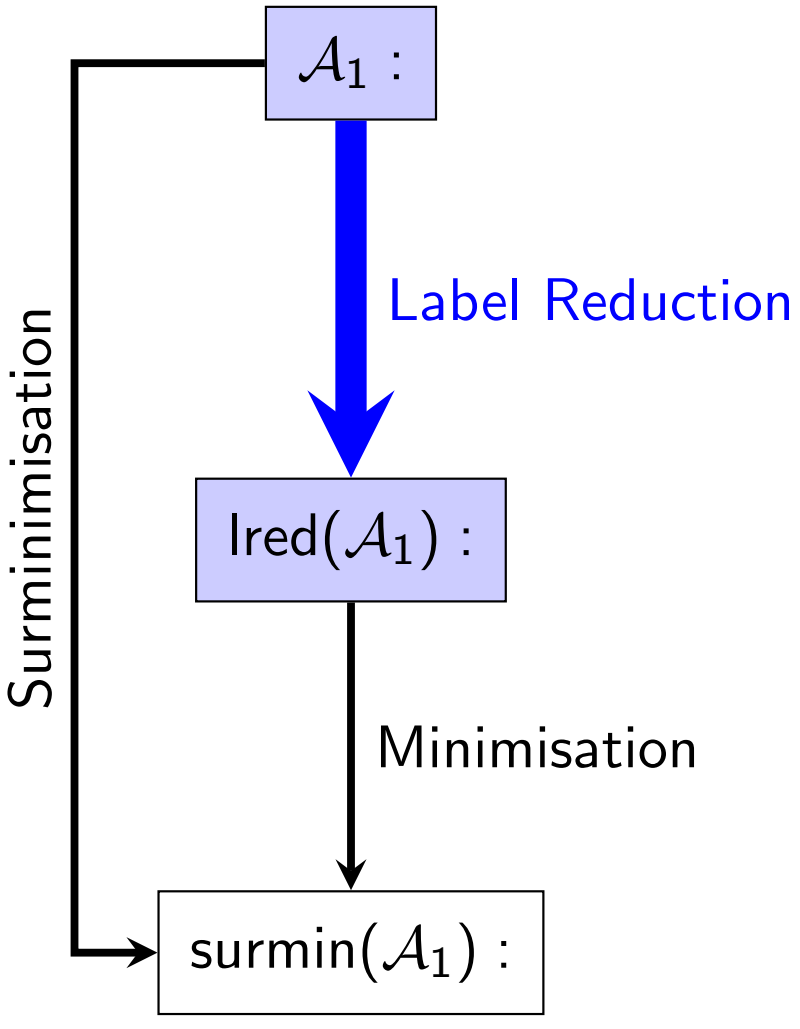

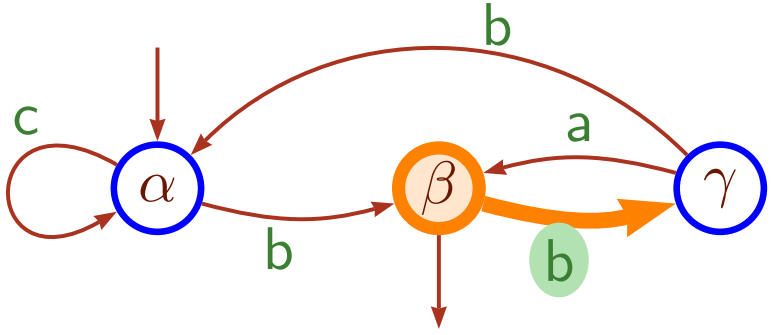

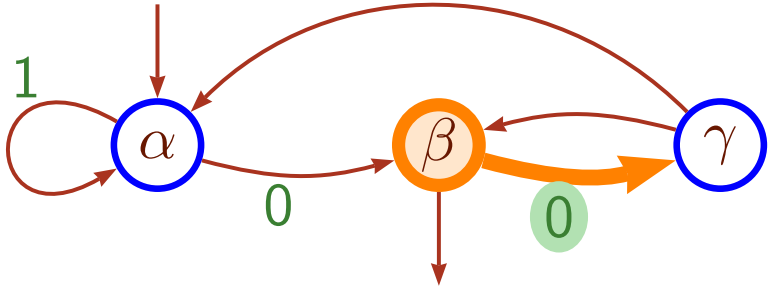

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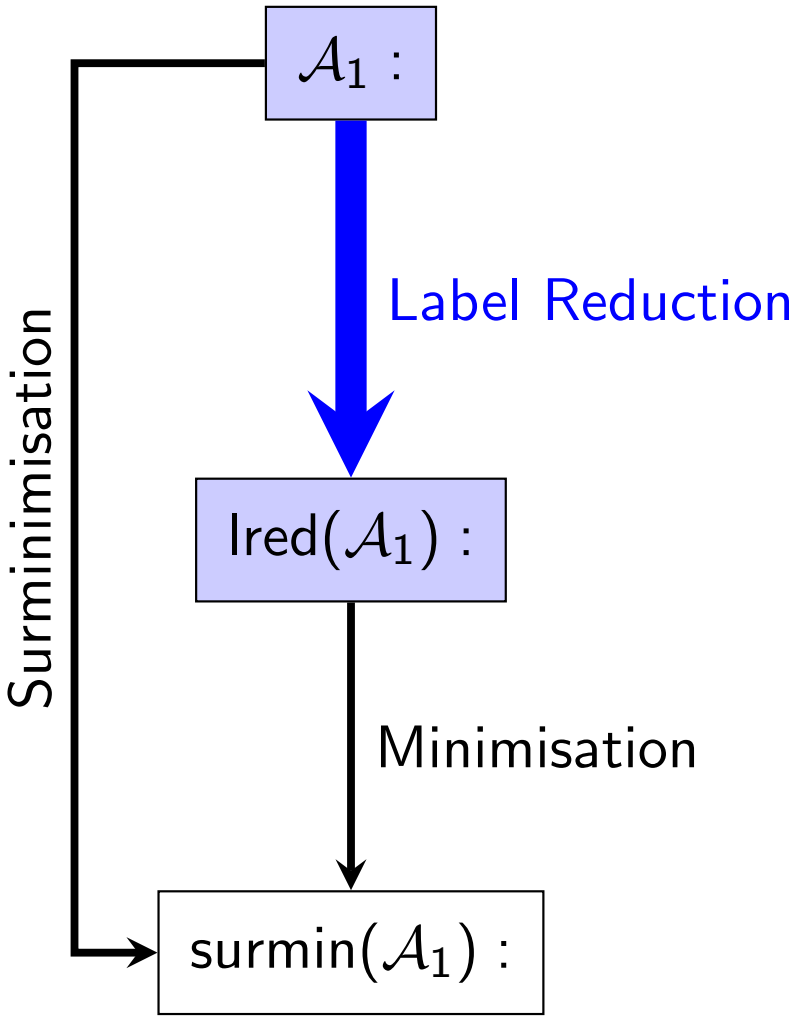

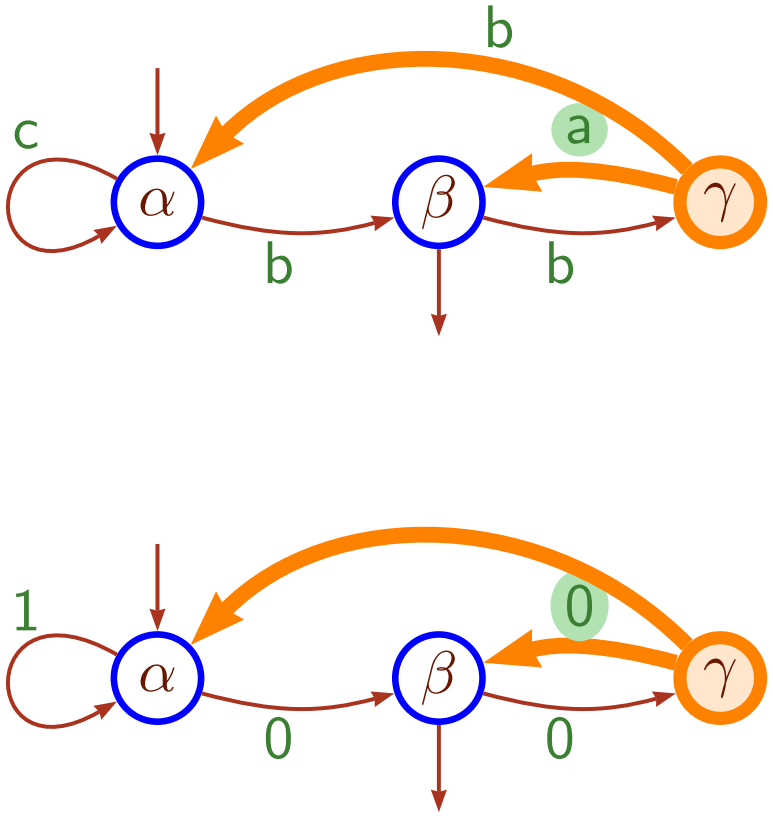

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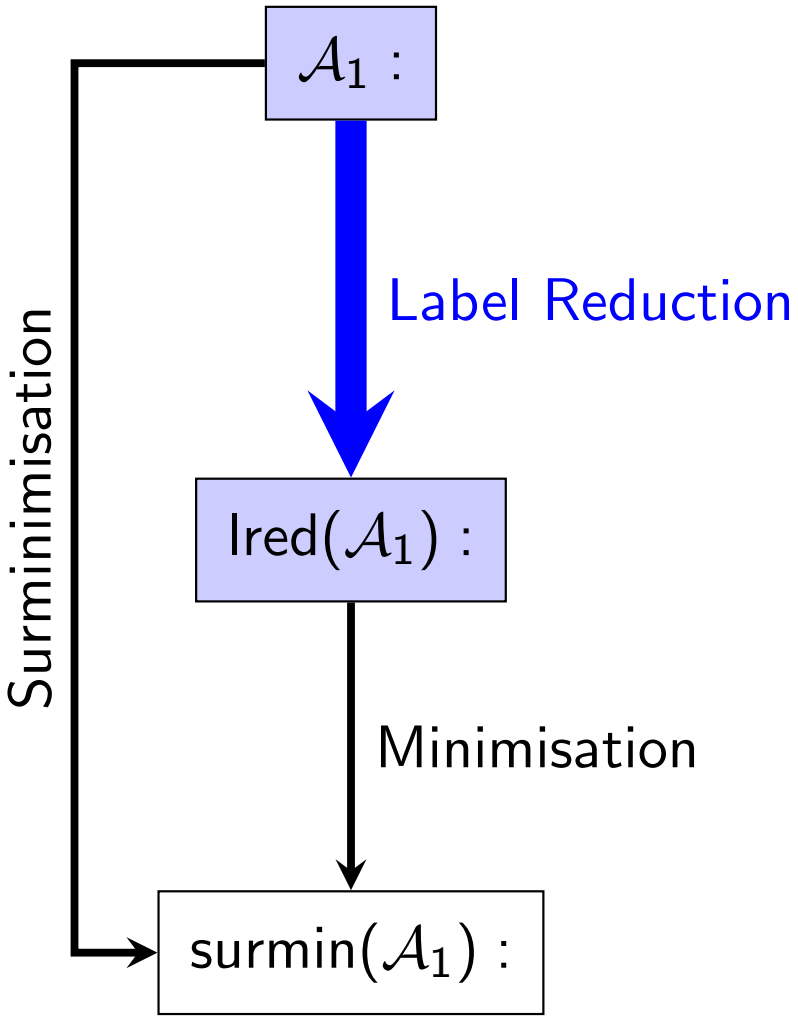

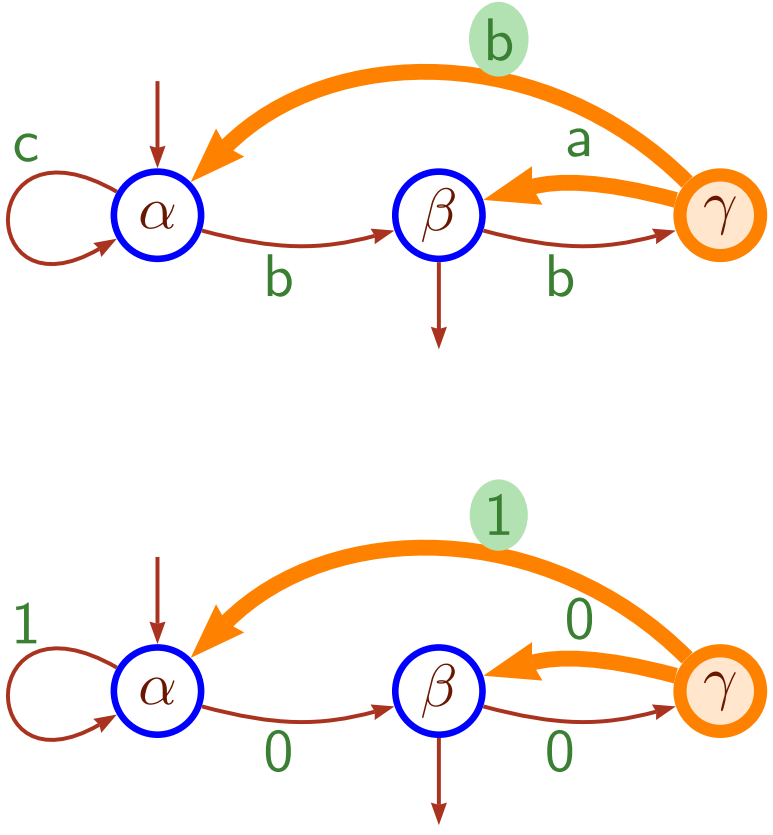

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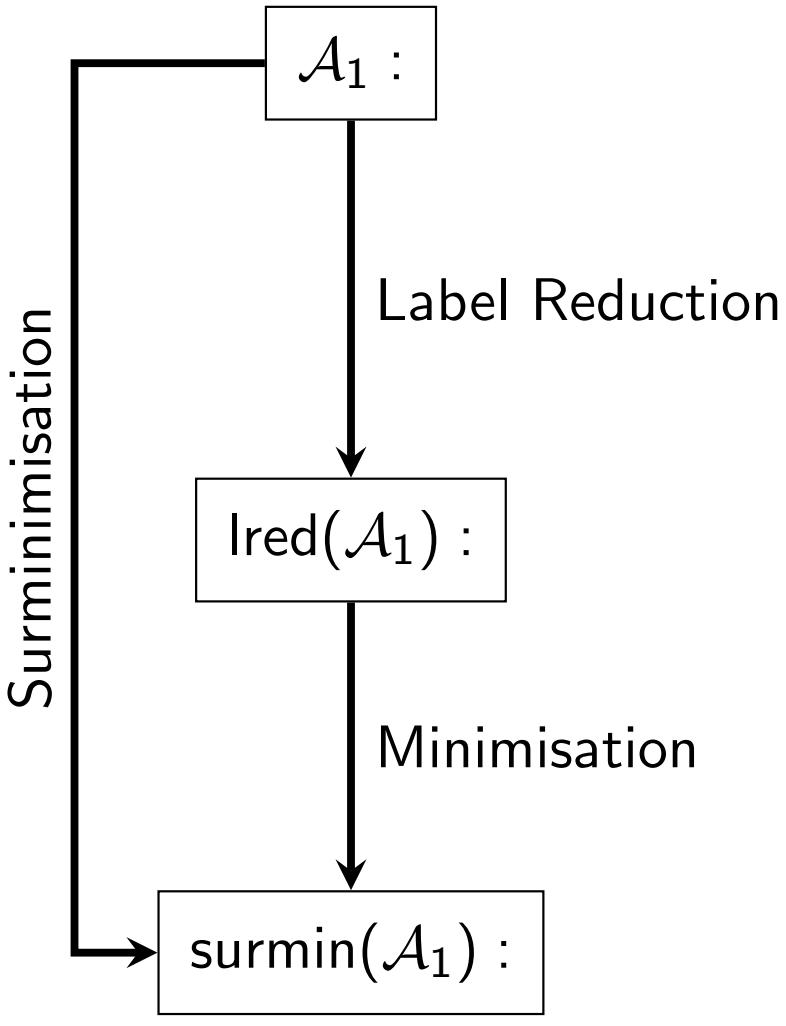

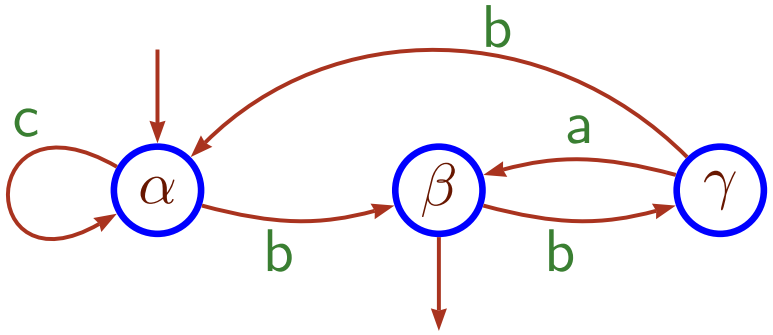

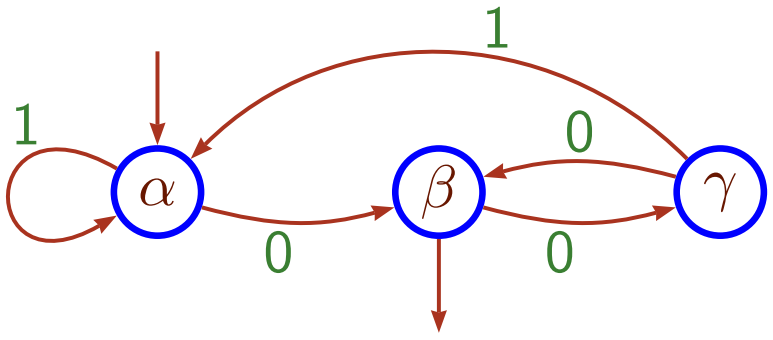

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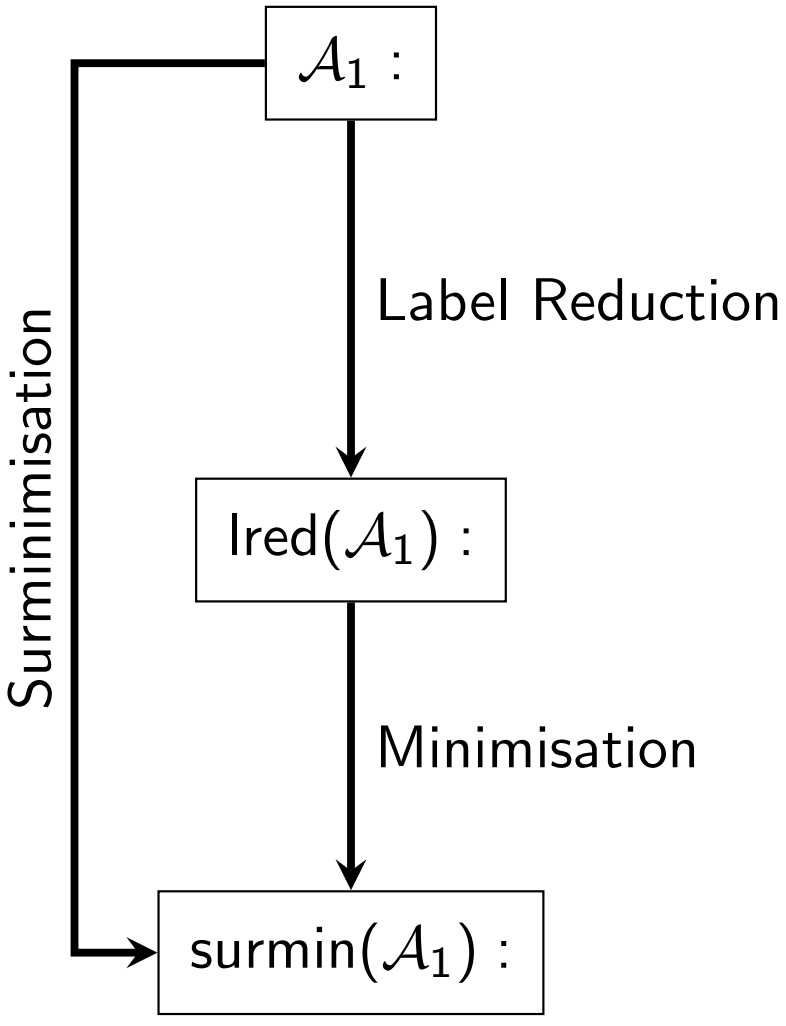

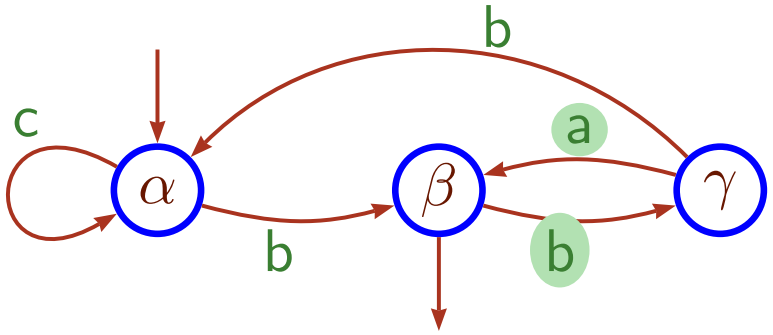

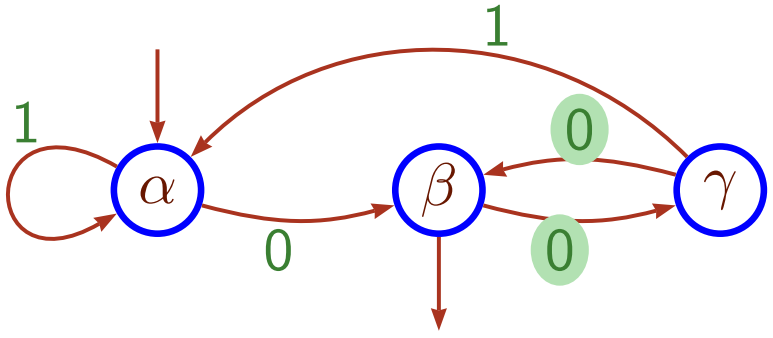

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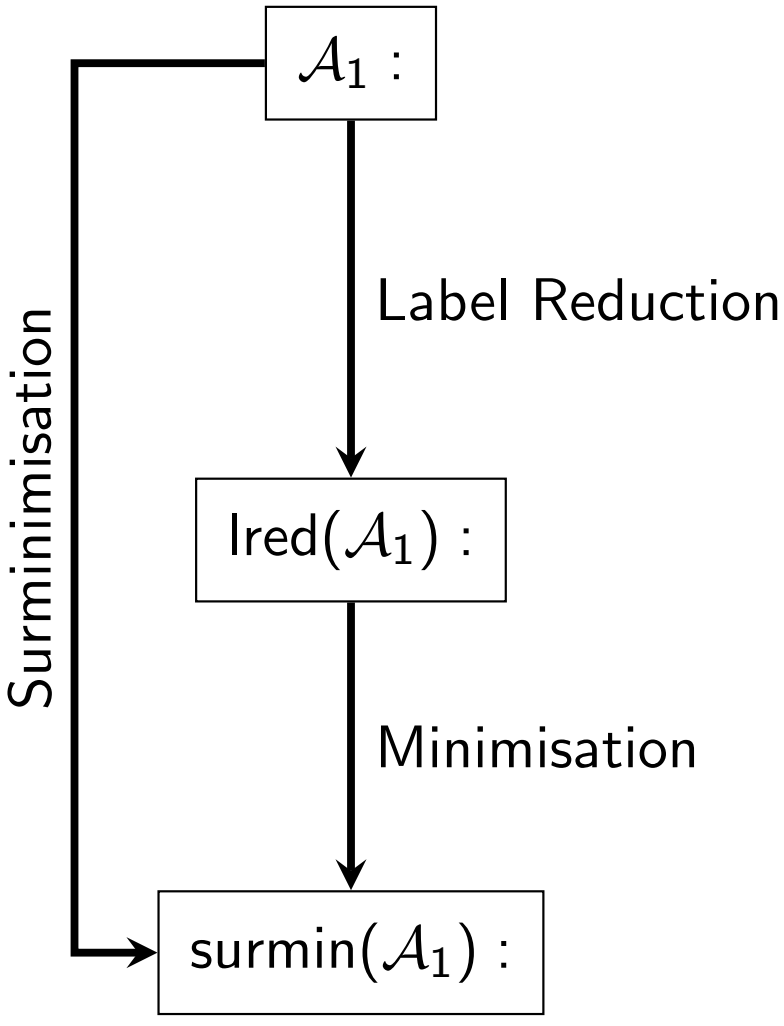

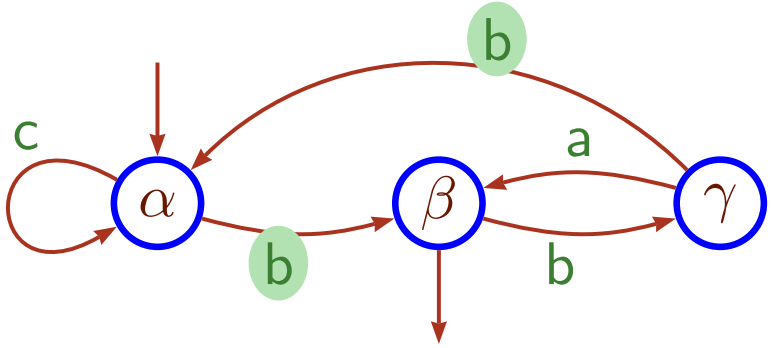

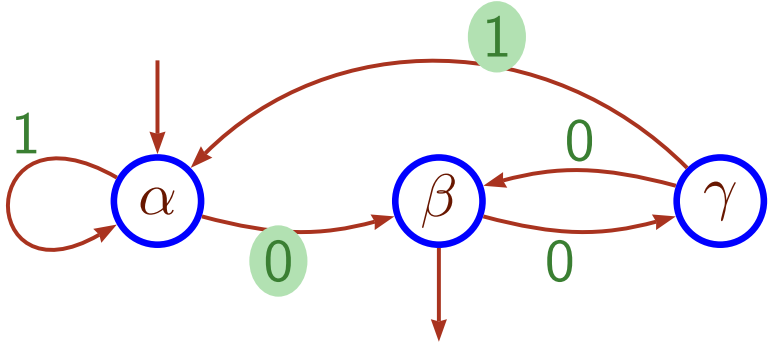

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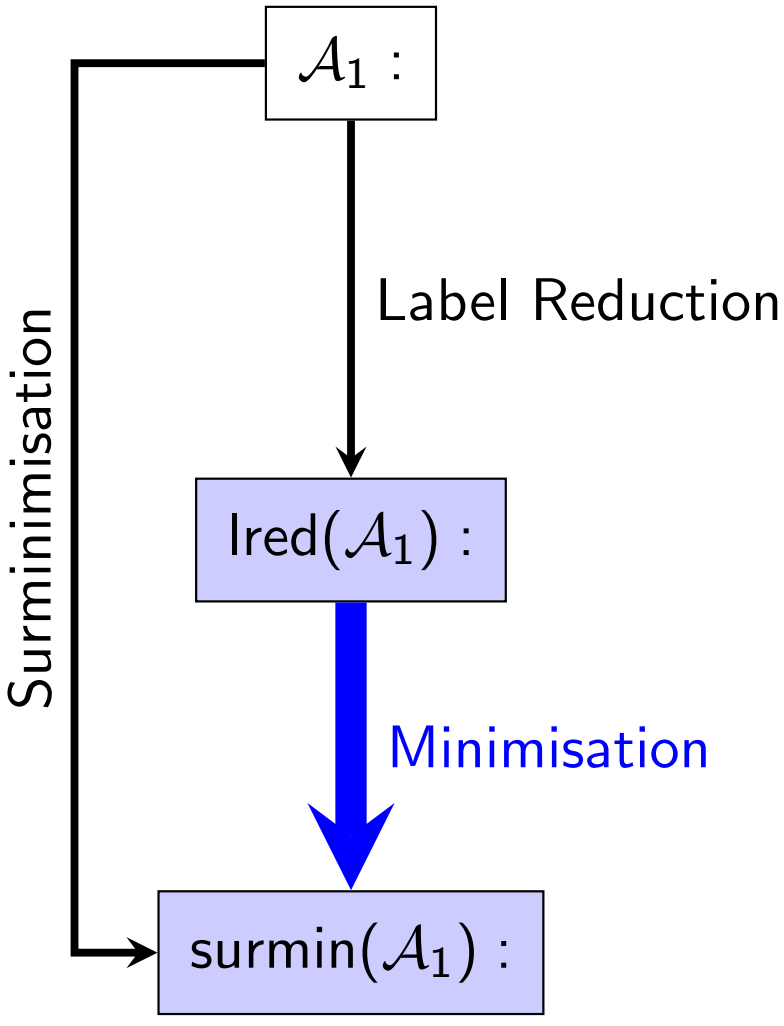
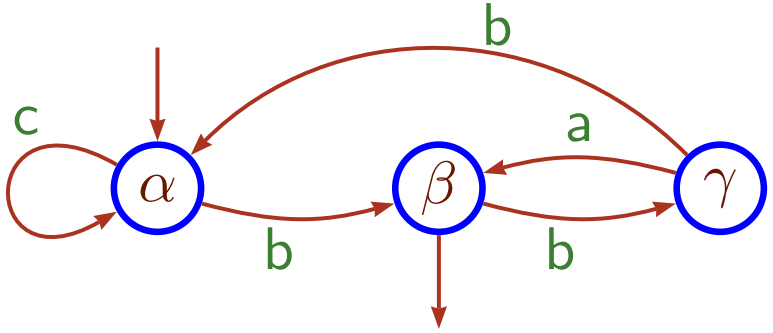
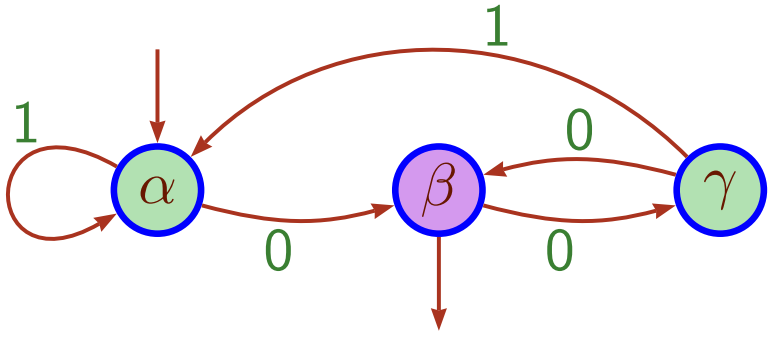
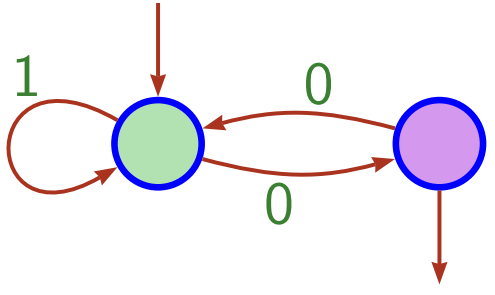


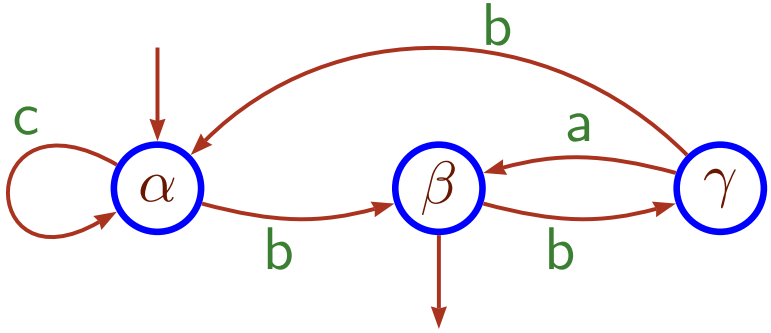






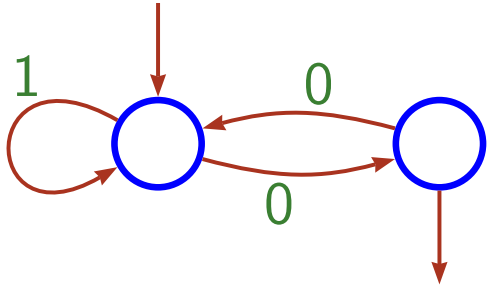
Example of Surminimisation (1)

$\mathcal{A}_1$:

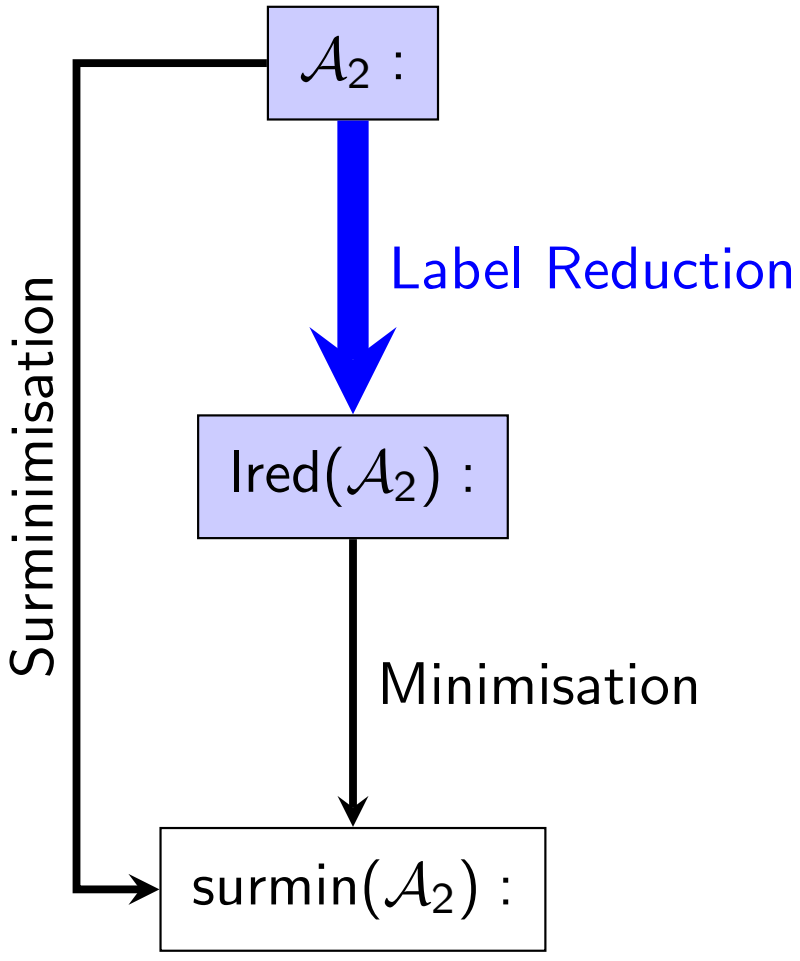


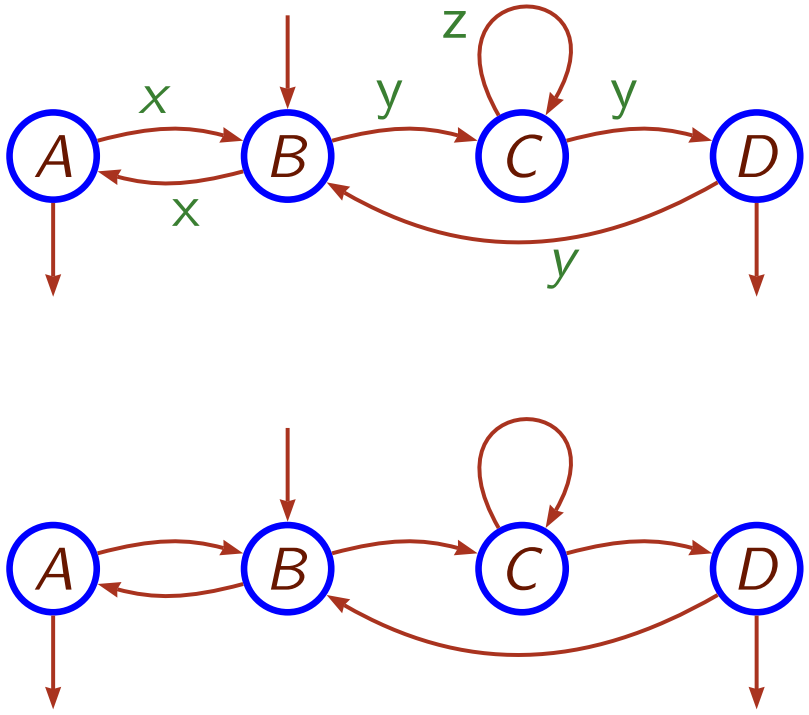
Surminimisation

surmin($\mathcal{A}_1$) :



Example of Surminimisation (2)





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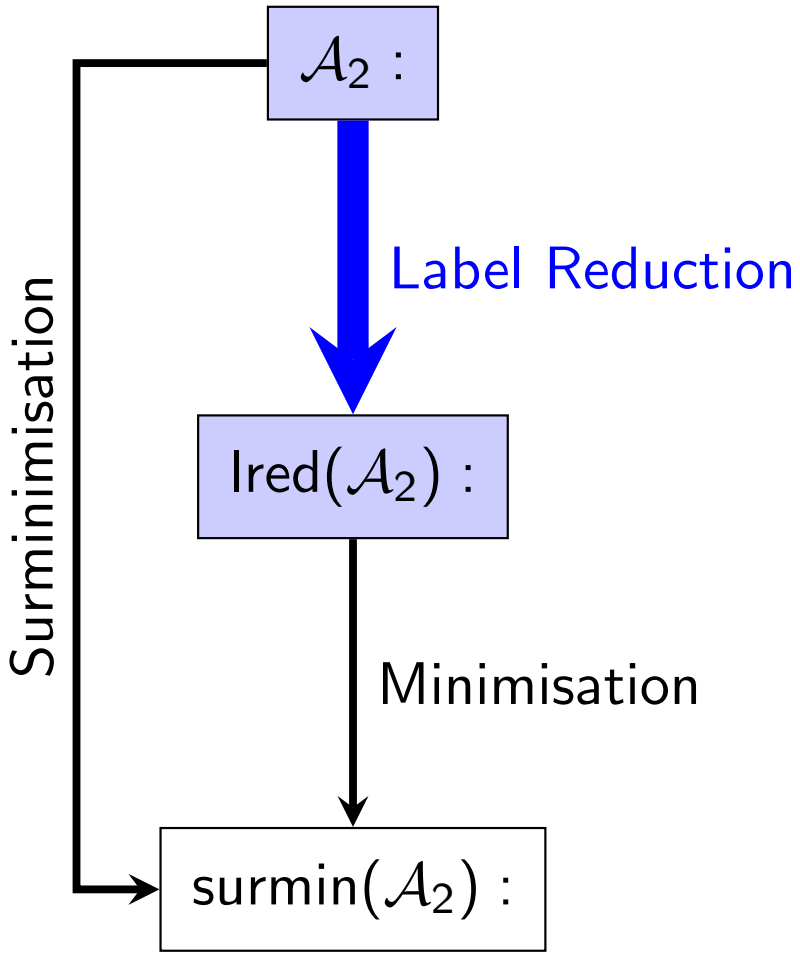

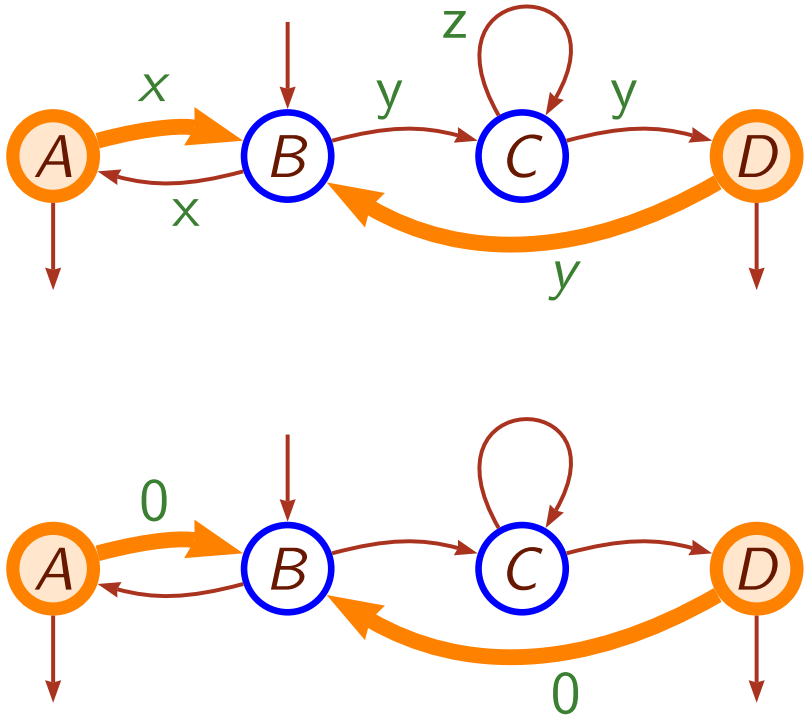

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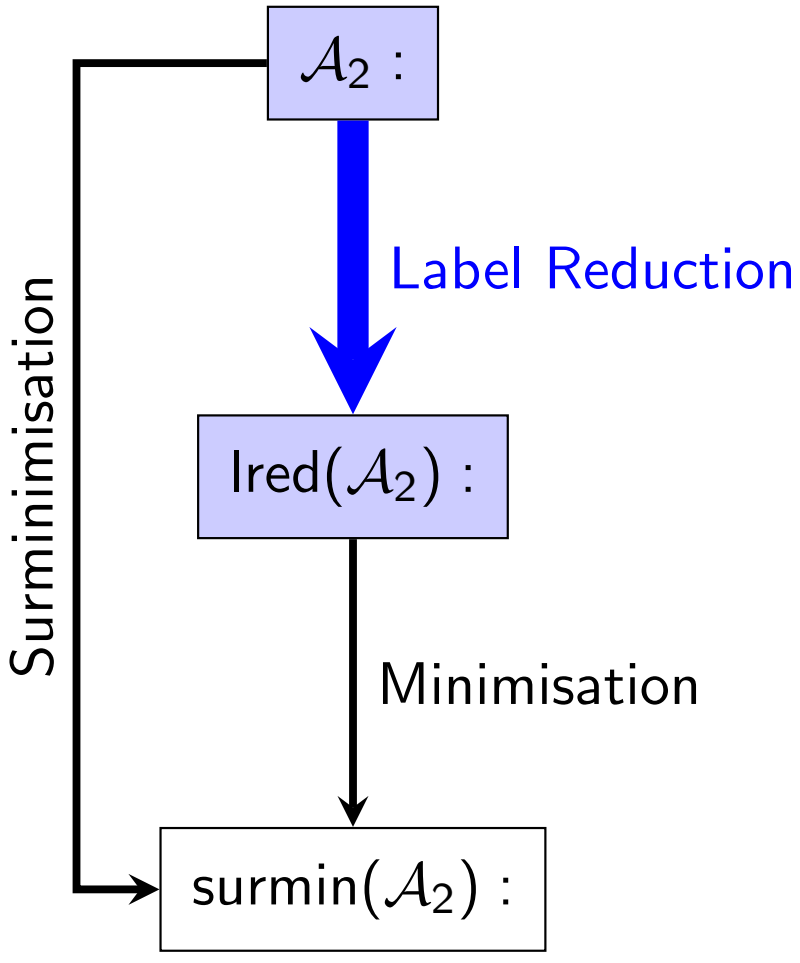

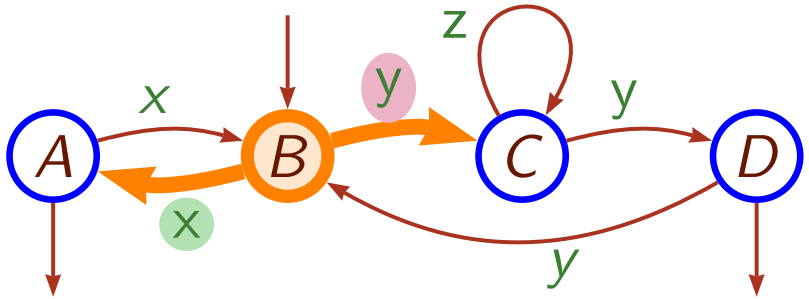

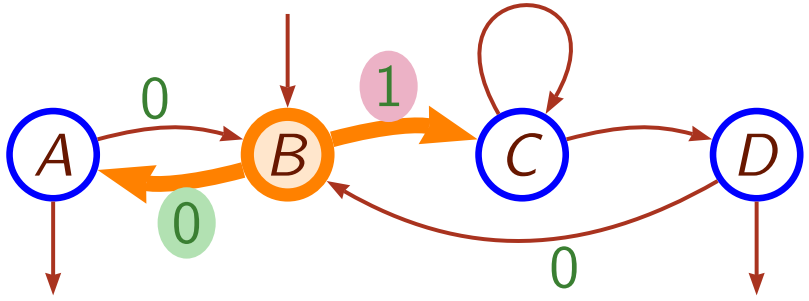

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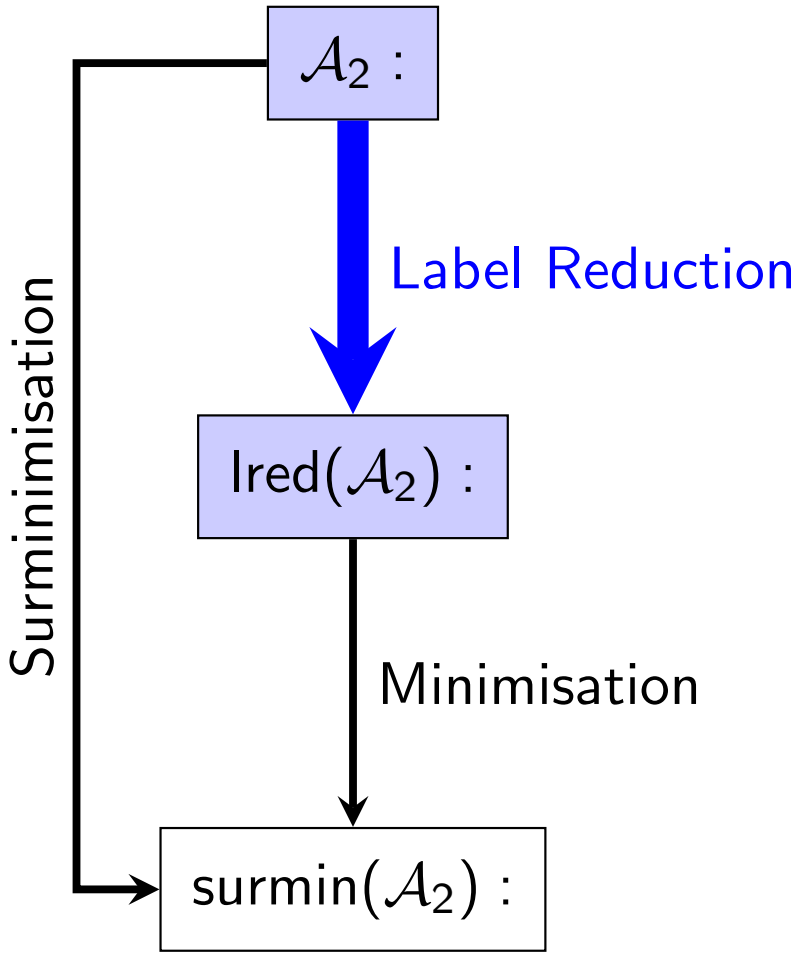

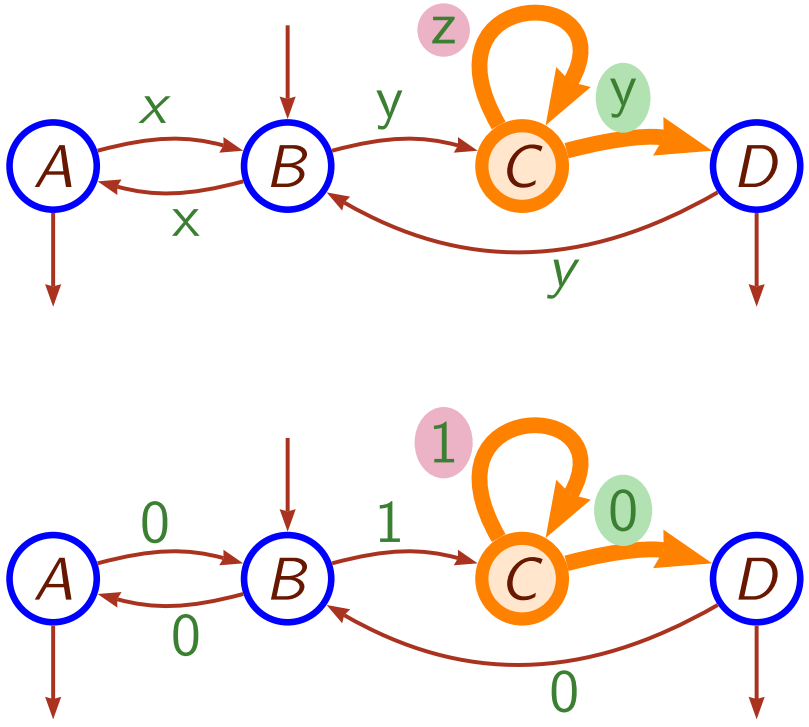

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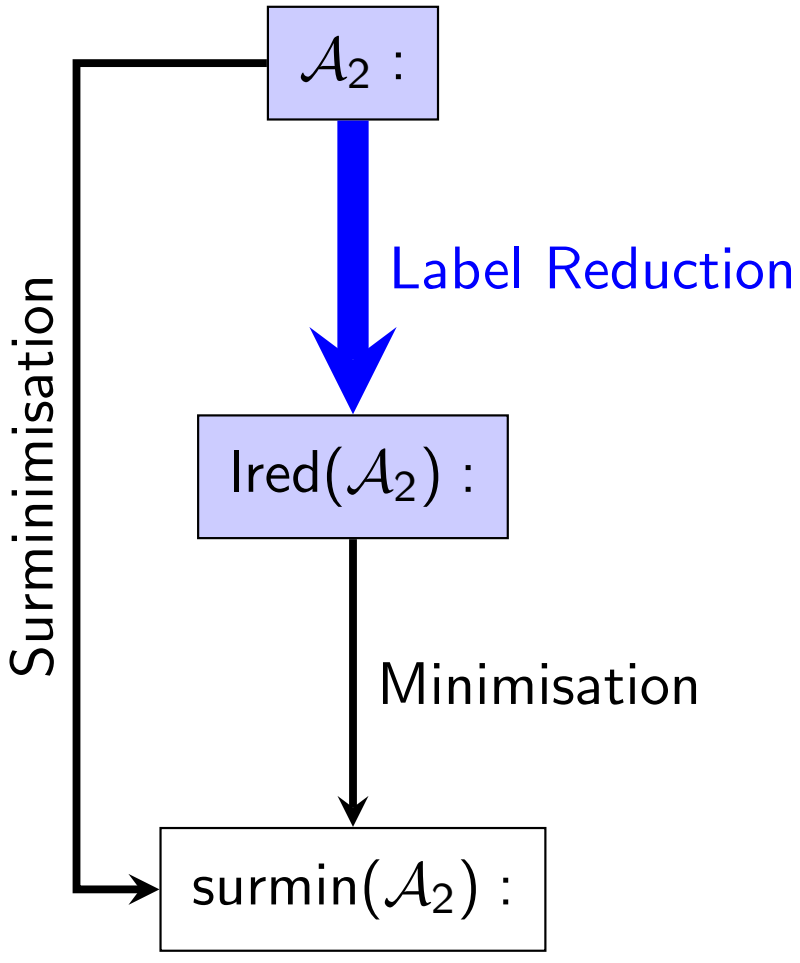

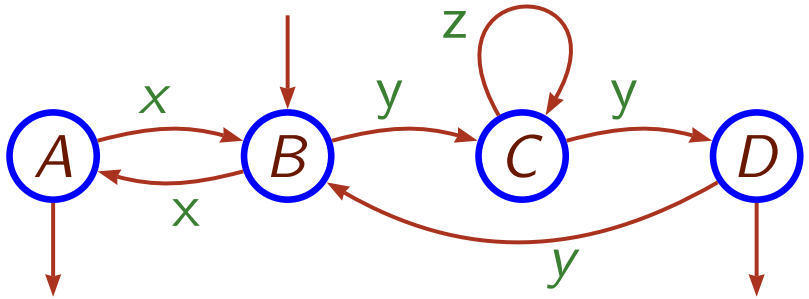

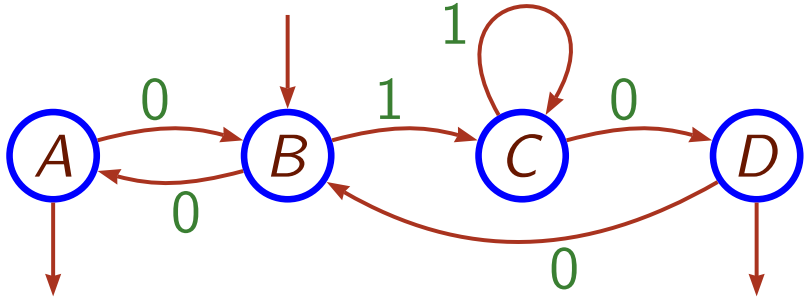

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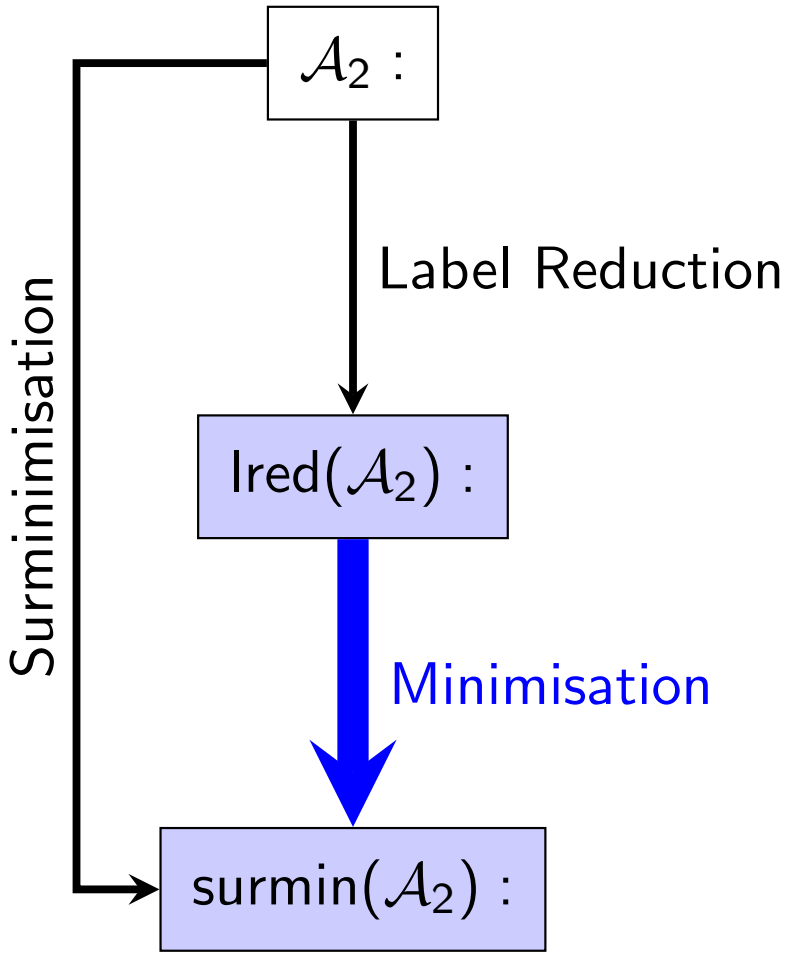
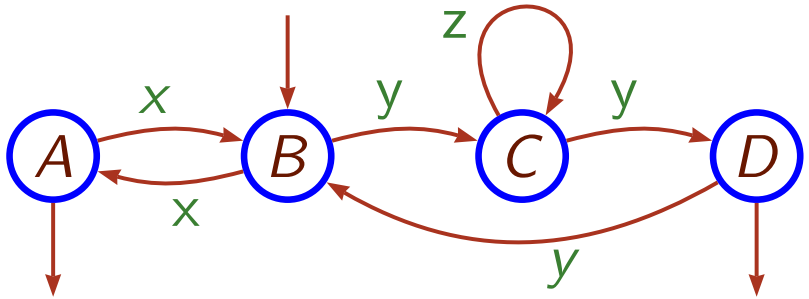
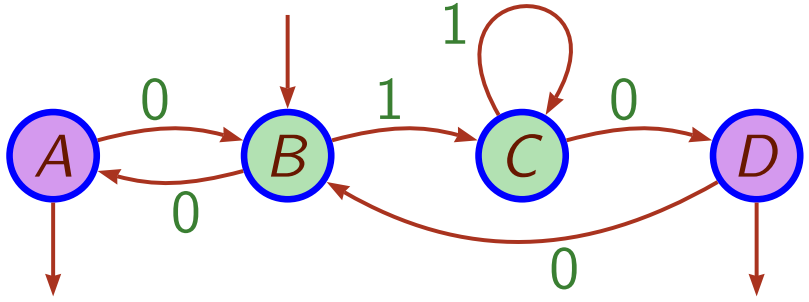
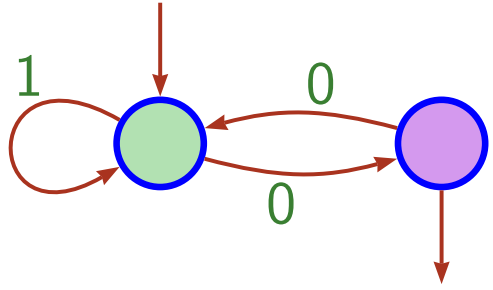


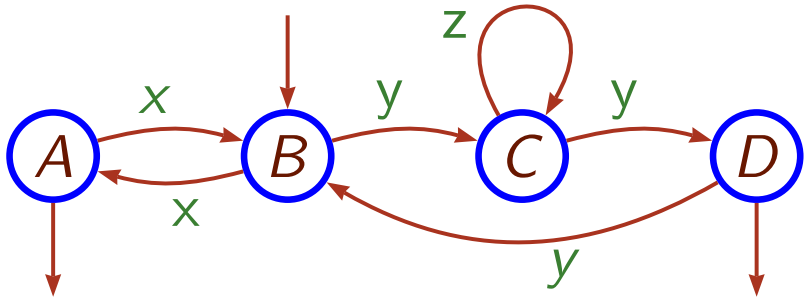






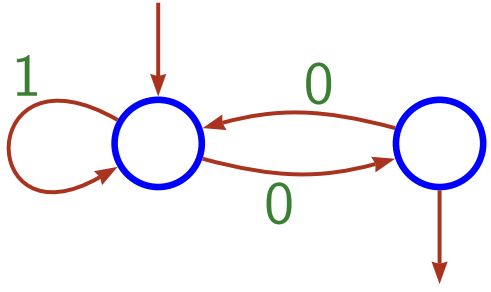
Example of Surminimisation (2)

$\mathcal{A}_2$:



Surminimisation

surmin($\mathcal{A}_2$) :



Properties of Label-Reduction

Lemma

Label-reduction is idempotent.

Commutation Lemma

Label-reduction commutes with automaton morphism.

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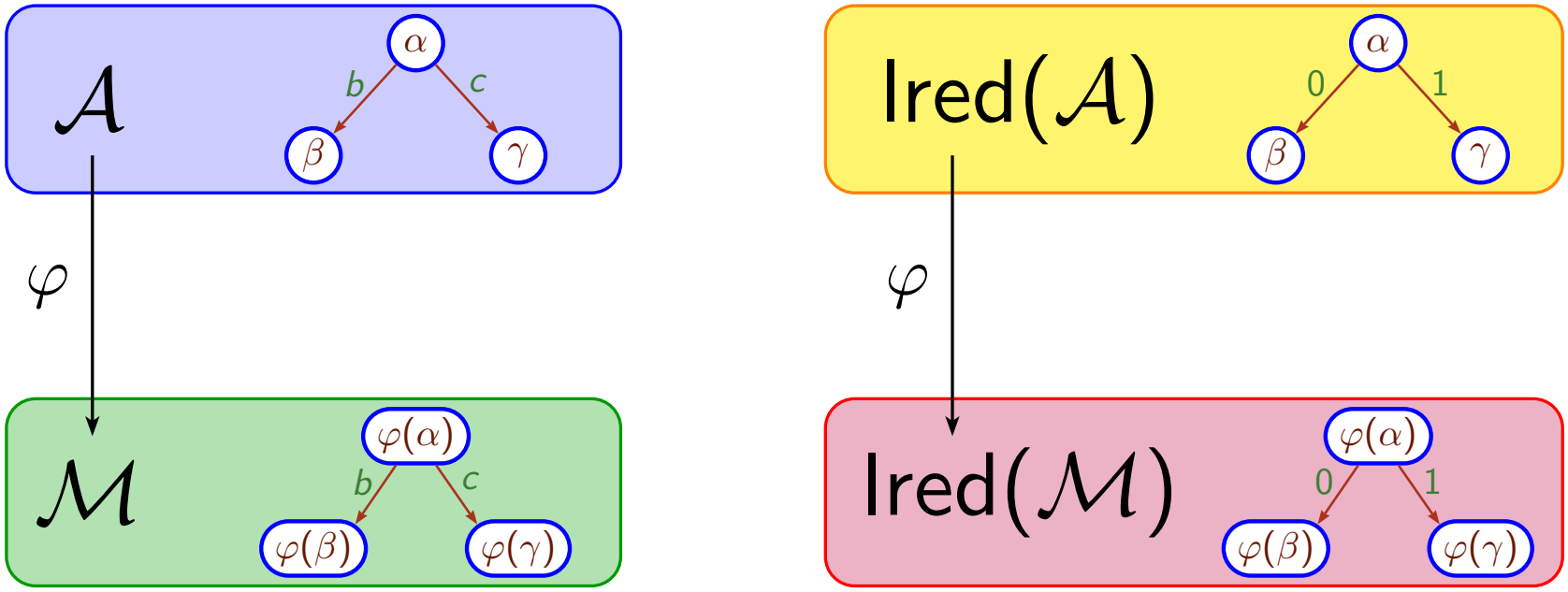

Remarks on Surminimisation

It removes the eventual meaning of letters

$L_3 = 0^*1^*$	*some 0's followed by some 1's*
$\mathsf{lred}(L_3) = 0^* + 0^*10^*$	*words with at most one 1*

In general, "marker" or "matching" symbols disapear.

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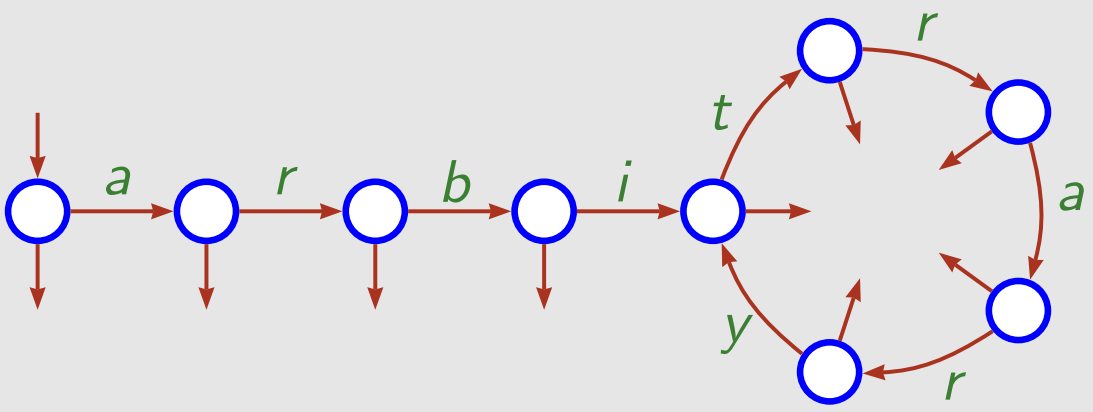
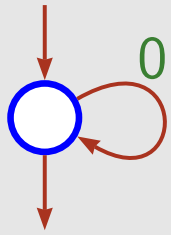


$\mathcal{A}_4$



$\mathrm{surmin}(\mathcal{A}_4)$

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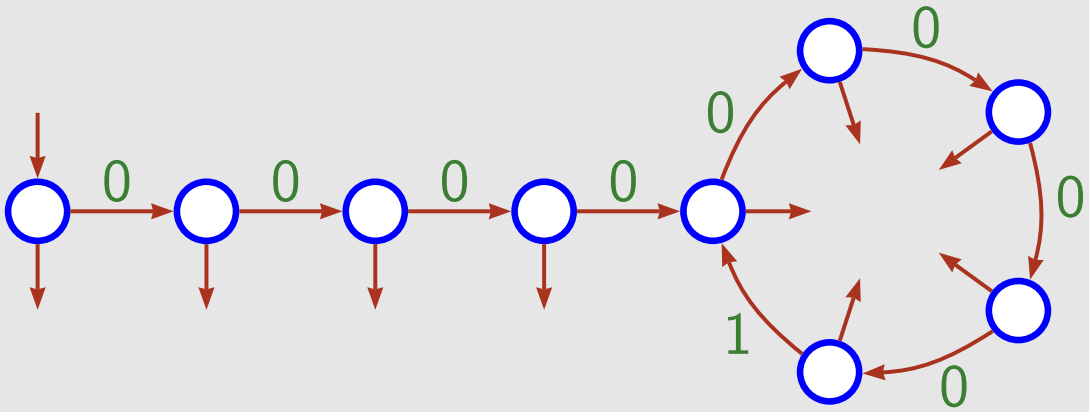

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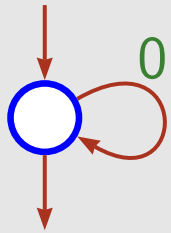


$\mathrm{surmin}(\mathcal{A}_4)$

Outline

T-equivalence (1)

Definition (T-equivalence)

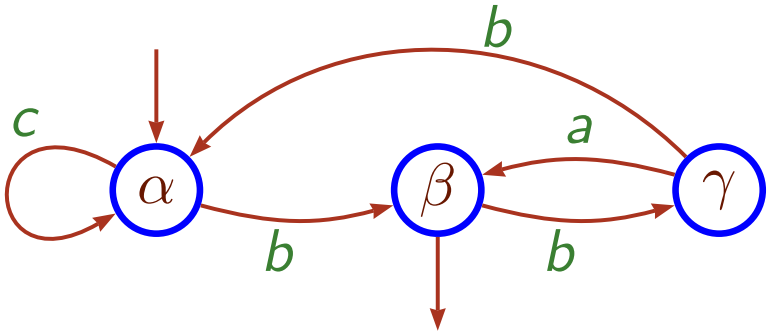
$\mathcal{A}$ and $\mathcal{B}$: two trim DFA

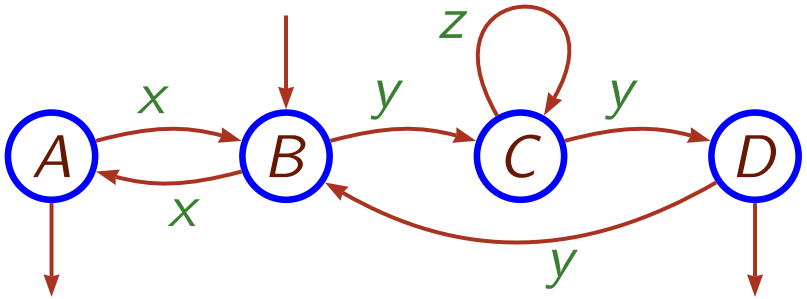
$\mathcal{A} \overset{T}{\sim} \mathcal{B}$ if $\mathsf{lred}(\mathcal{A})$ and $\mathsf{lred}(\mathcal{B})$ accept the same language

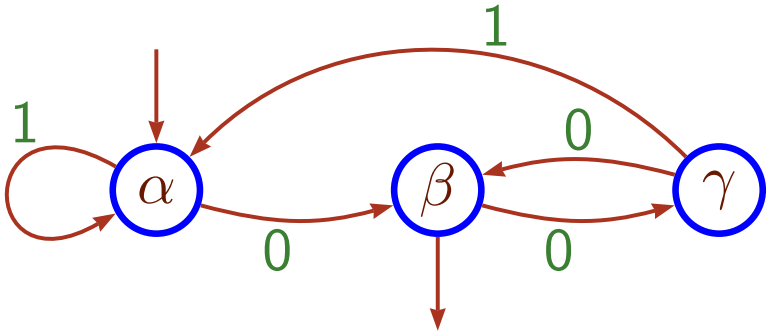
Equivalent definition

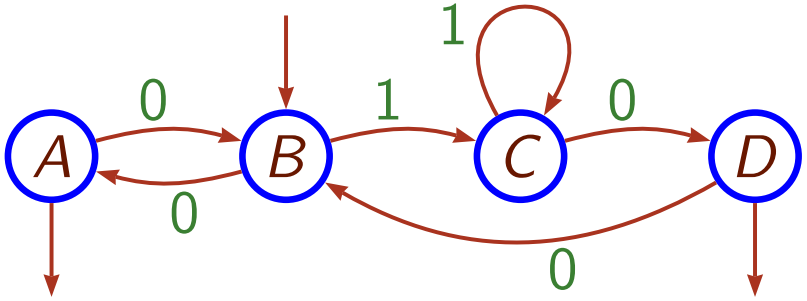
$$\mathcal{A} \overset{T}{\sim} \mathcal{B} \iff \mathsf{surmin}(\mathcal{A}) = \mathsf{surmin}(\mathcal{B})$$

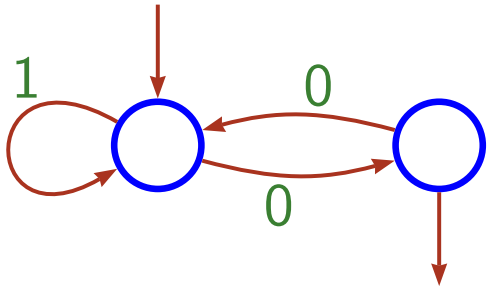
$\mathcal{A}_1$ and $\mathcal{A}_2$ are T-equivalent











T-equivalence (2)

Proposition

$\mathcal{A}$ and $\mathcal{B}$: two **trim** automata.

$$L(\mathcal{A}) = L(\mathcal{B}) \quad \Longrightarrow \quad L(\mathsf{Ired}(\mathcal{A})) = L(\mathsf{Ired}(\mathcal{B}))$$

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Proof.
Let $\mathcal{M}$ be the minimisation of $\mathcal{A}$ and $\mathcal{B}$.
$\Longrightarrow$ $\exists$ Two morphisms $\phi : \mathcal{A} \to \mathcal{M}$ and $\psi : \mathcal{B} \to \mathcal{M}$.
Apply Commutation Lemma.

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Corollary

T-equivalence is a coarser relation than equivalence.

† Two automata are equivalent if they accept the same language.

Representative of T-equivalence class

Proposition

Surminimisation is idempotent.

(Follows from the *commutation Lemma* and the idempotence of label-reduction)

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Each T-equivalence class has a canonical representative.

Invariant of T-equivalence

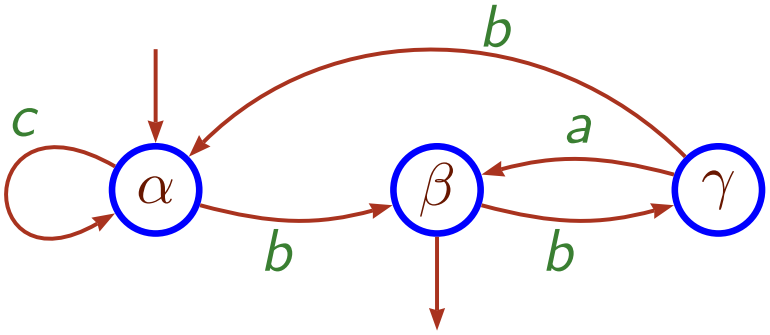
Definition (Automaton Unfolding)

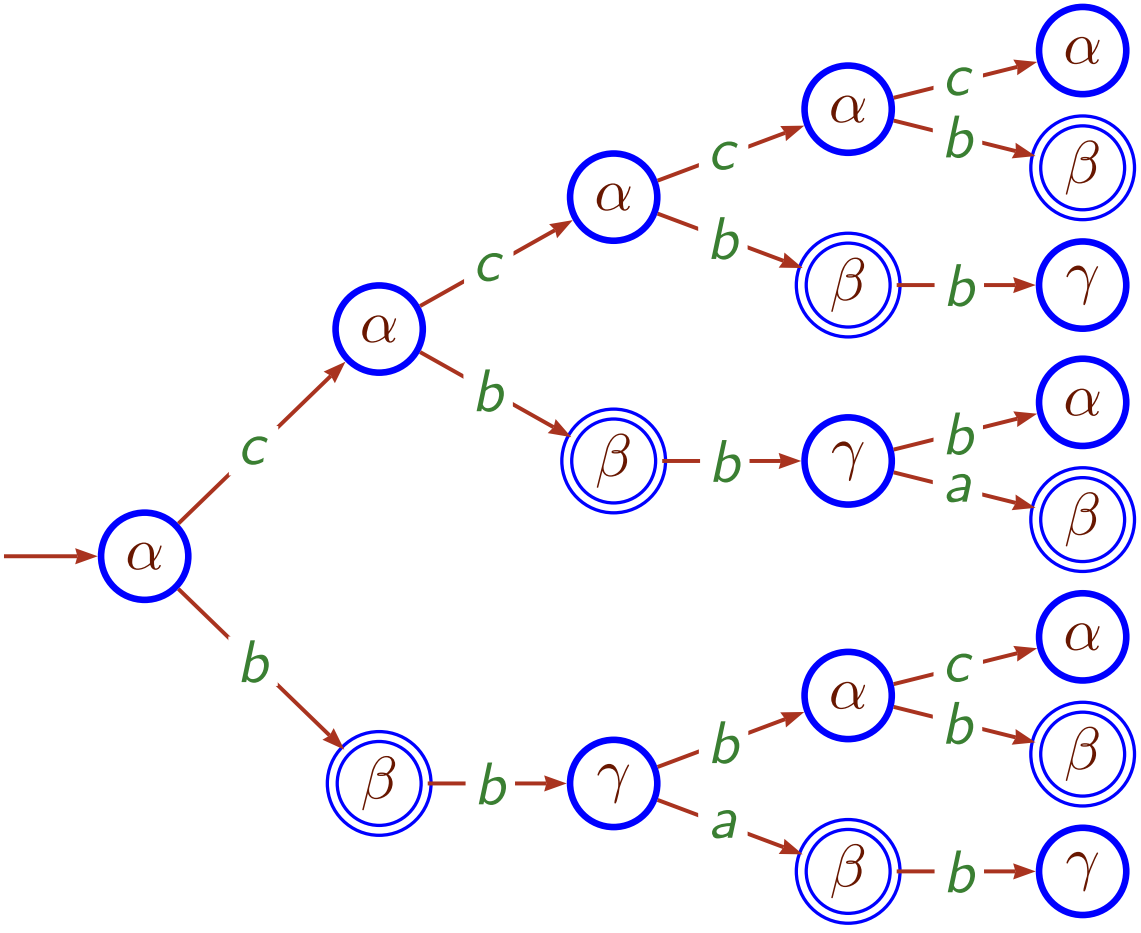
A DFA $\mathcal{A} \xrightarrow{\text{unfolds into}}$ a labelled tree $T_{\mathcal{A}}$.

Runs of $\mathcal{A} \xrightarrow{\text{becomes}}$ branches of $T_{\mathcal{A}}$.

When $\mathcal{A}$ is trim, $T_{\mathcal{A}}$ is also the labelled tree representing $L(\mathcal{A})$.

Invariant of T-equivalence – Unfolding of $\mathcal{A}_1$





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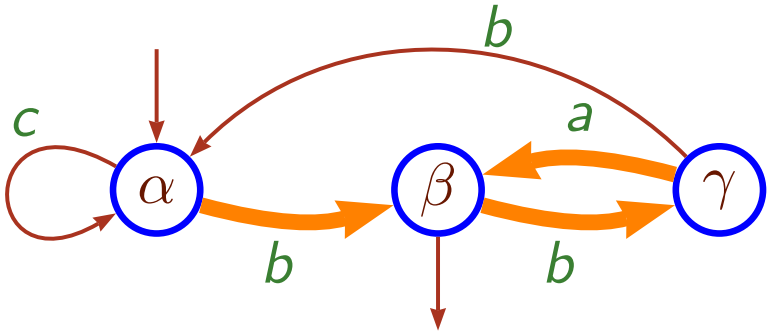

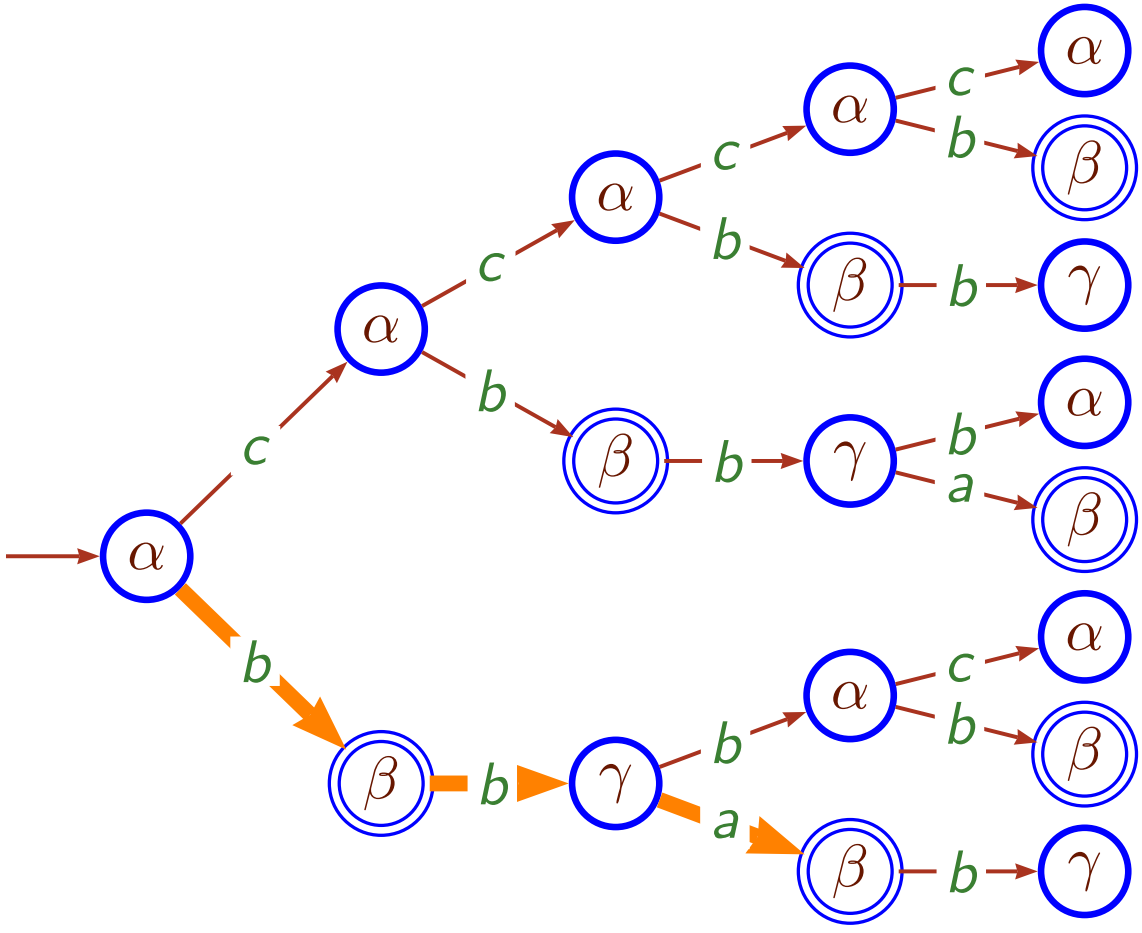

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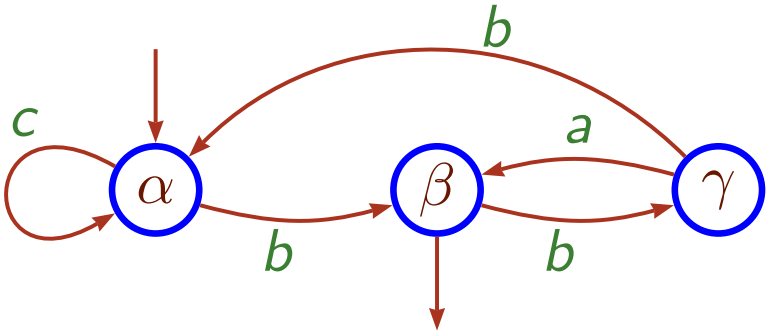
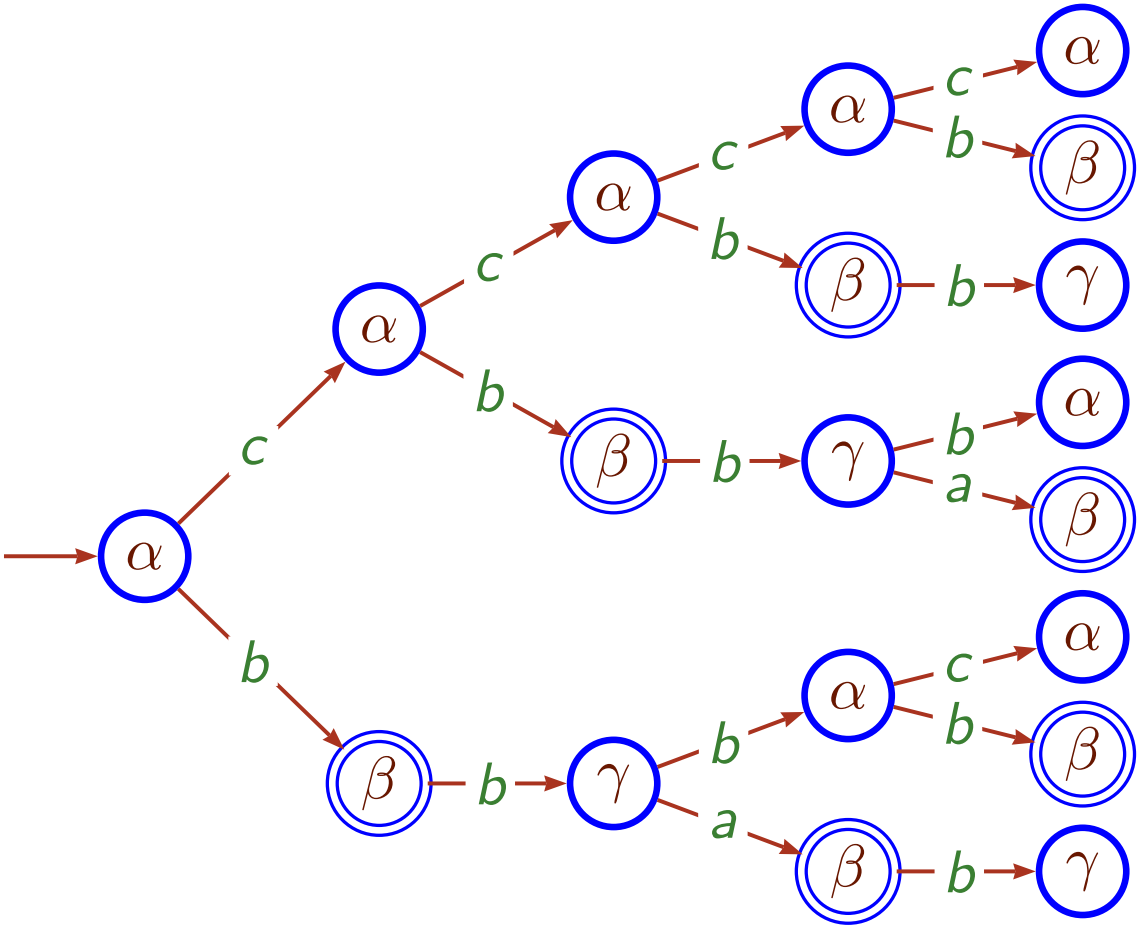
Proposition

$\mathcal{A}$, $\mathcal{B}$: two trim DFA.

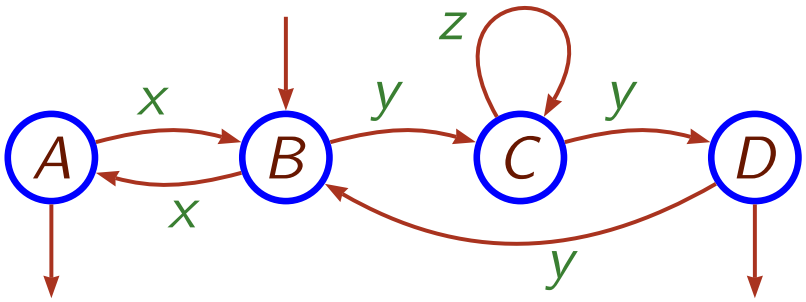
$$\mathcal{A} \overset{T}{\sim} \mathcal{B} \iff T_{\mathcal{A}} \text{ and } T_{\mathcal{B}} \text{ differ only by labelling}$$

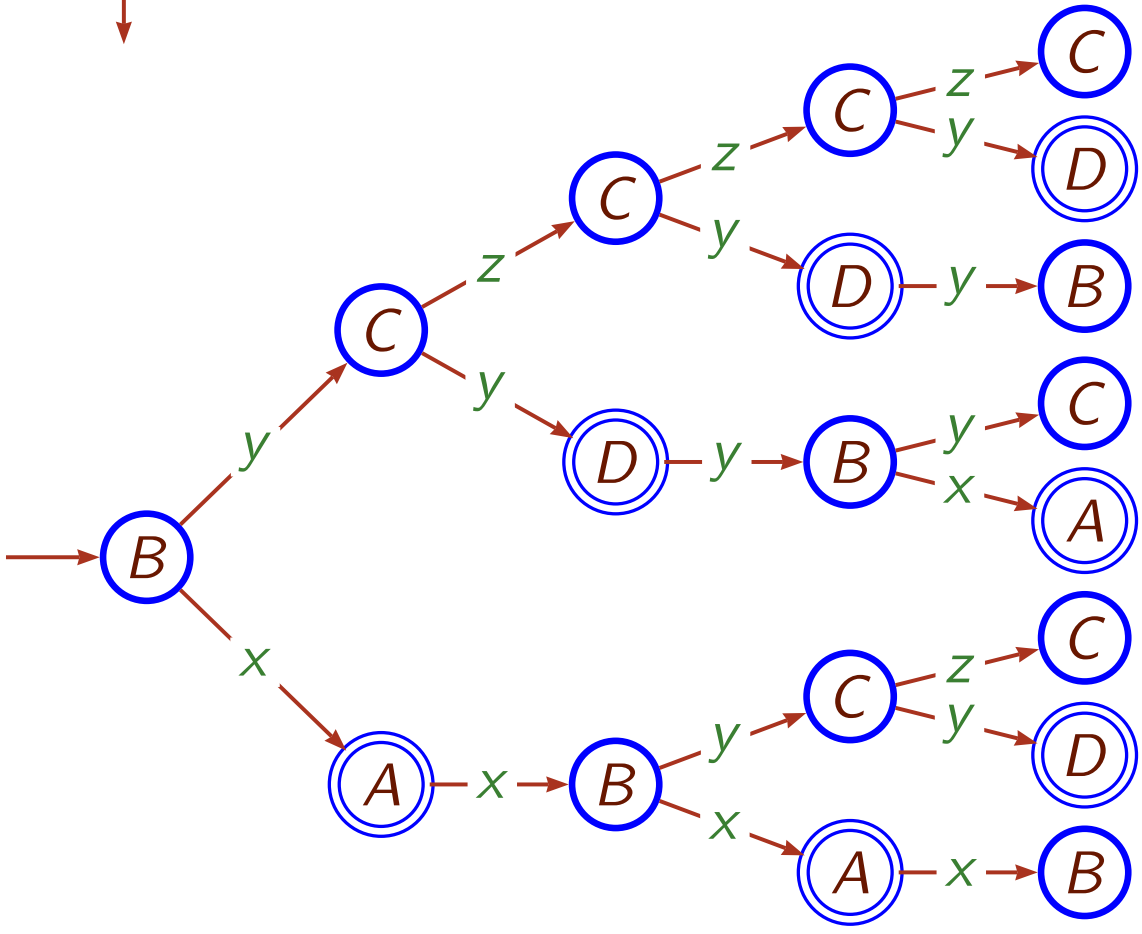
Invariant of T-equivalence – Unfolding of $\mathcal{A}_1$





Invariant of T-equivalence – Unfolding of $\mathcal{A}_2$





Outline

Ordering A^*

Definition (Radix order)

A: an alphabet provided with a total order.

A^* is ordered by the *radix order*:

$$u <_{\text{rad}} v \quad \textit{if} \quad \begin{cases} |u| < |v| \\ \quad \textit{or} \\ |u| = |v| \text{ and } u \text{ is lexicograph. smaller than } v \end{cases}$$

Example : $\epsilon <_{\text{rad}} 20 <_{\text{rad}} 21 <_{\text{rad}} 100$

Abstract Regular Numeration Systems

Definition (ARNS, Lecomte-Rigo 2001)

L: a regular language over A.

An integer n is represented by the word $\langle n \rangle_L$ (in the ARNS L) where $\langle n \rangle_L$ is the $(n+1)$-th word of L (in the radix order).

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Example

$L_3 = a^* b^*$
Enumeration of L_3 according to the radix order:

ϵ	a	b	aa	ab	bb	aaa	aab	abb	bbb	$\cdots$
$\langle 0 \rangle$	$\langle 1 \rangle$	$\langle 2 \rangle$	$\langle 3 \rangle$	$\langle 4 \rangle$	$\langle 5 \rangle$	$\langle 6 \rangle$	$\langle 7 \rangle$	$\langle 8 \rangle$	$\langle 9 \rangle$	$\cdots$

Theorem

L, K: two ARNS accepted by two DFA $\mathcal{A}$ and $\mathcal{B}$

$\mathcal{A} \overset{T}{\sim} \mathcal{B} \implies \exists$ Mealy Machine† $\mathcal{A} \boxtimes \mathcal{B}$ realising $\langle n \rangle_L \mapsto \langle n \rangle_K$

† Letter-to-letter and pure sequential transducer.

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$\mathcal{A} \boxtimes \mathcal{B}$ is basically $\mathrm{Ired}(\mathcal{A}) \times \mathrm{Ired}(\mathcal{B})$ where
$(p, p') \xrightarrow{i} (q, q')$ is relabelled by $(p, p') \xrightarrow{a|x} (q, q')$ if

- $p \xrightarrow{a} q$ is the $(i+1)$-th transition going out of p in $\mathcal{A}$.
- $p' \xrightarrow{x} q'$ is the $(i+1)$-th transition going out of p' in $\mathcal{B}$.

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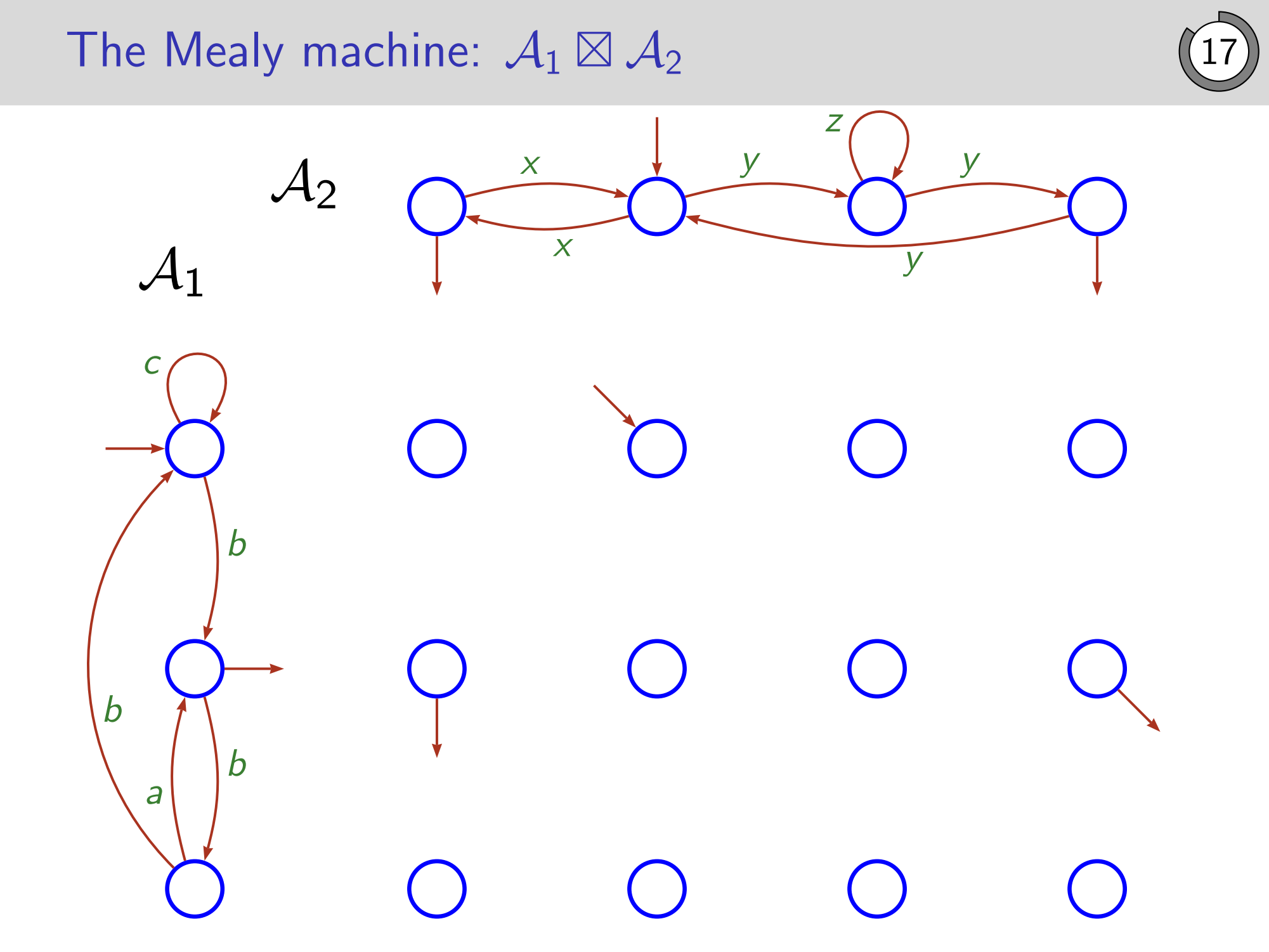
The Mealy machine: $\mathcal{A}_1 \boxtimes \mathcal{A}_2$



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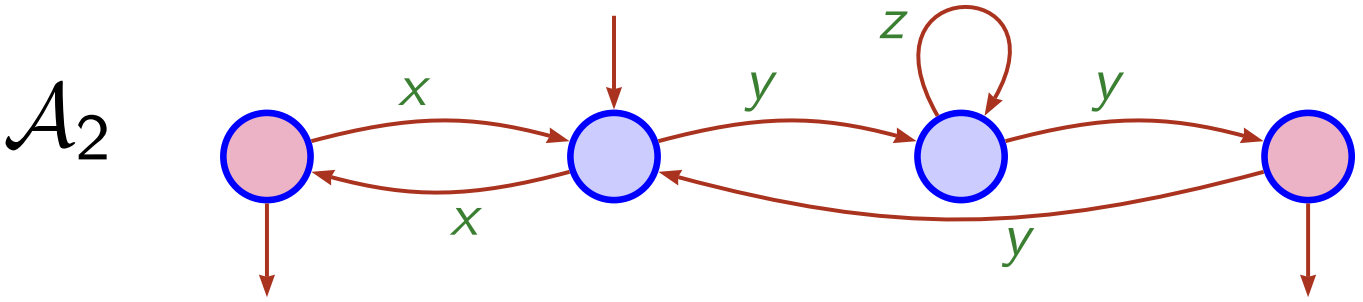

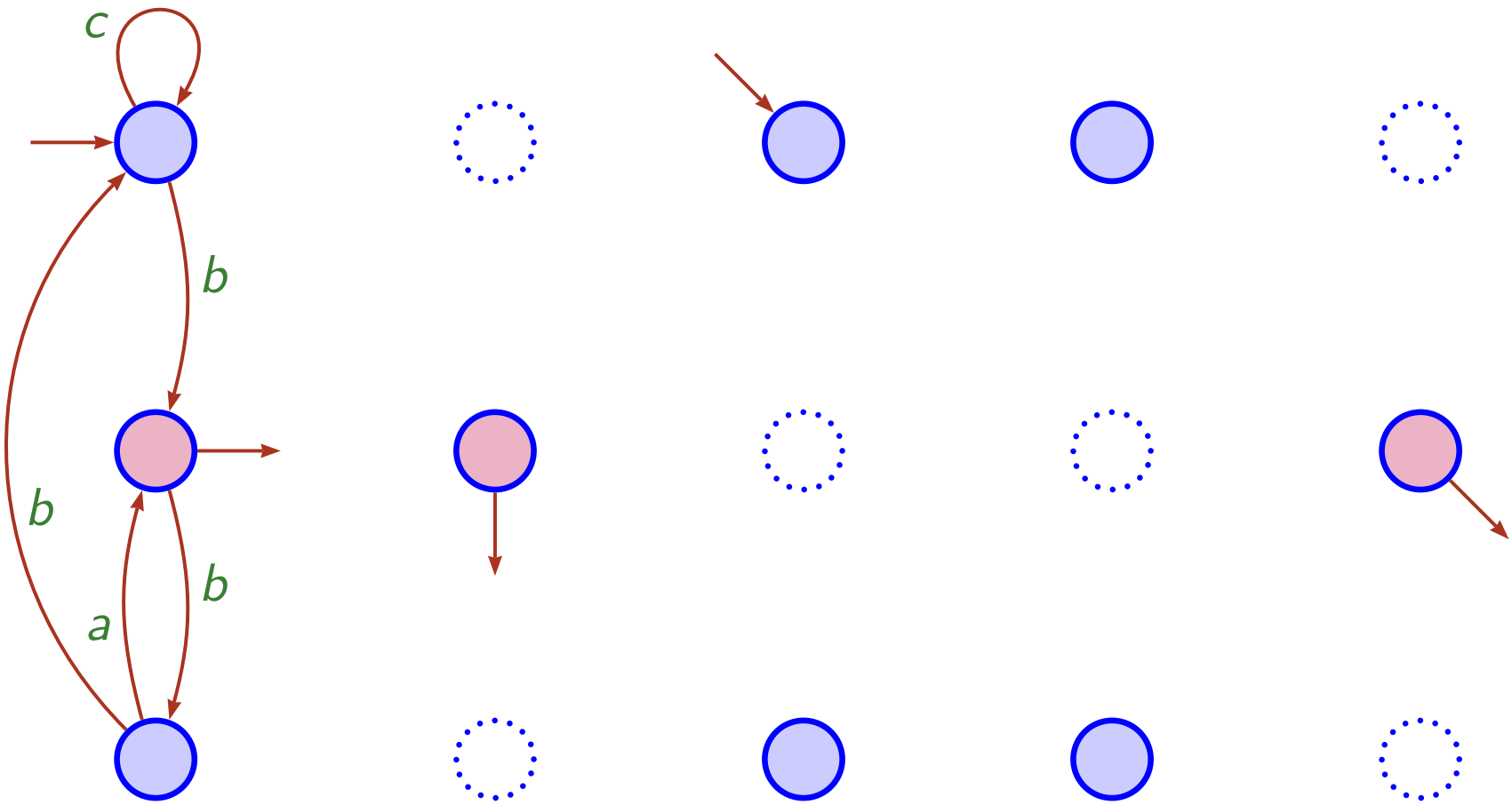

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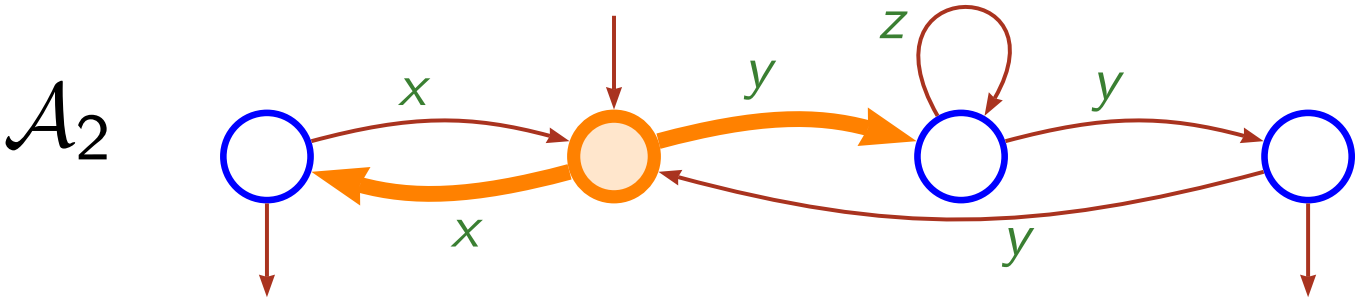

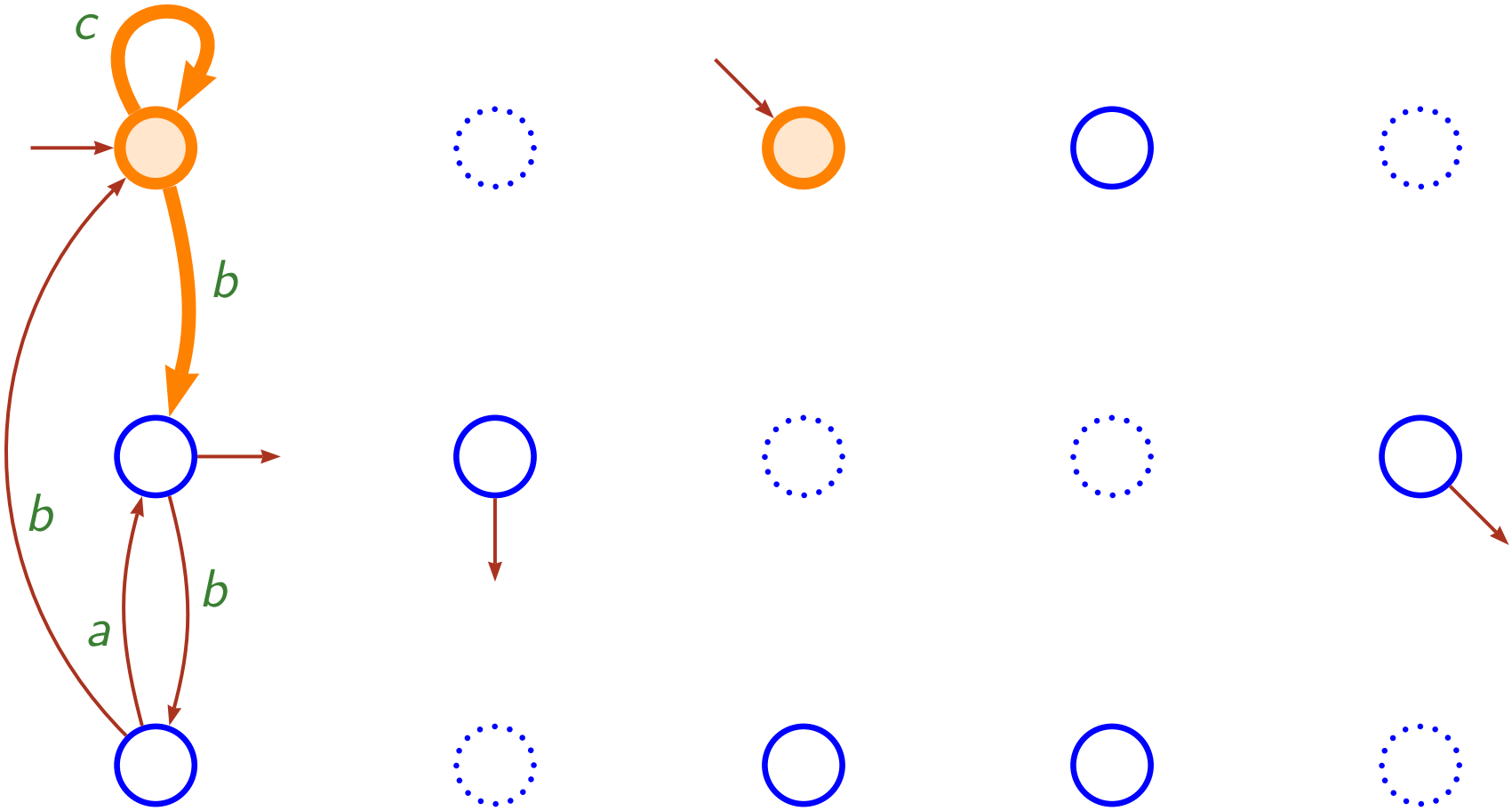

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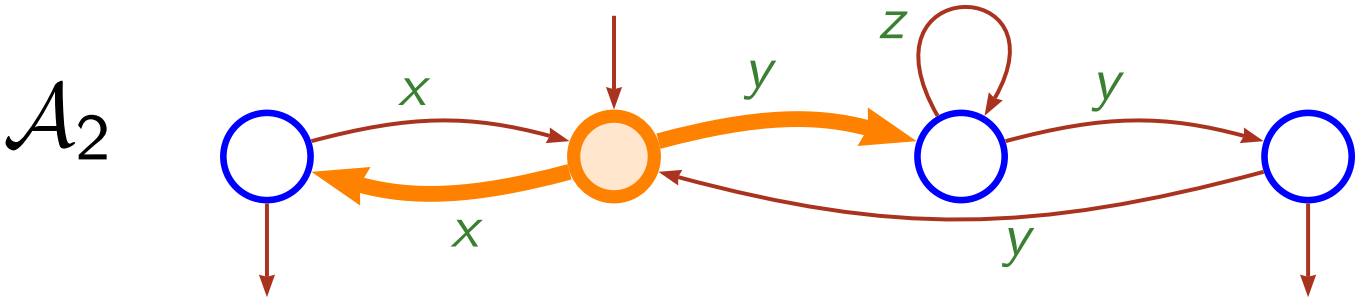

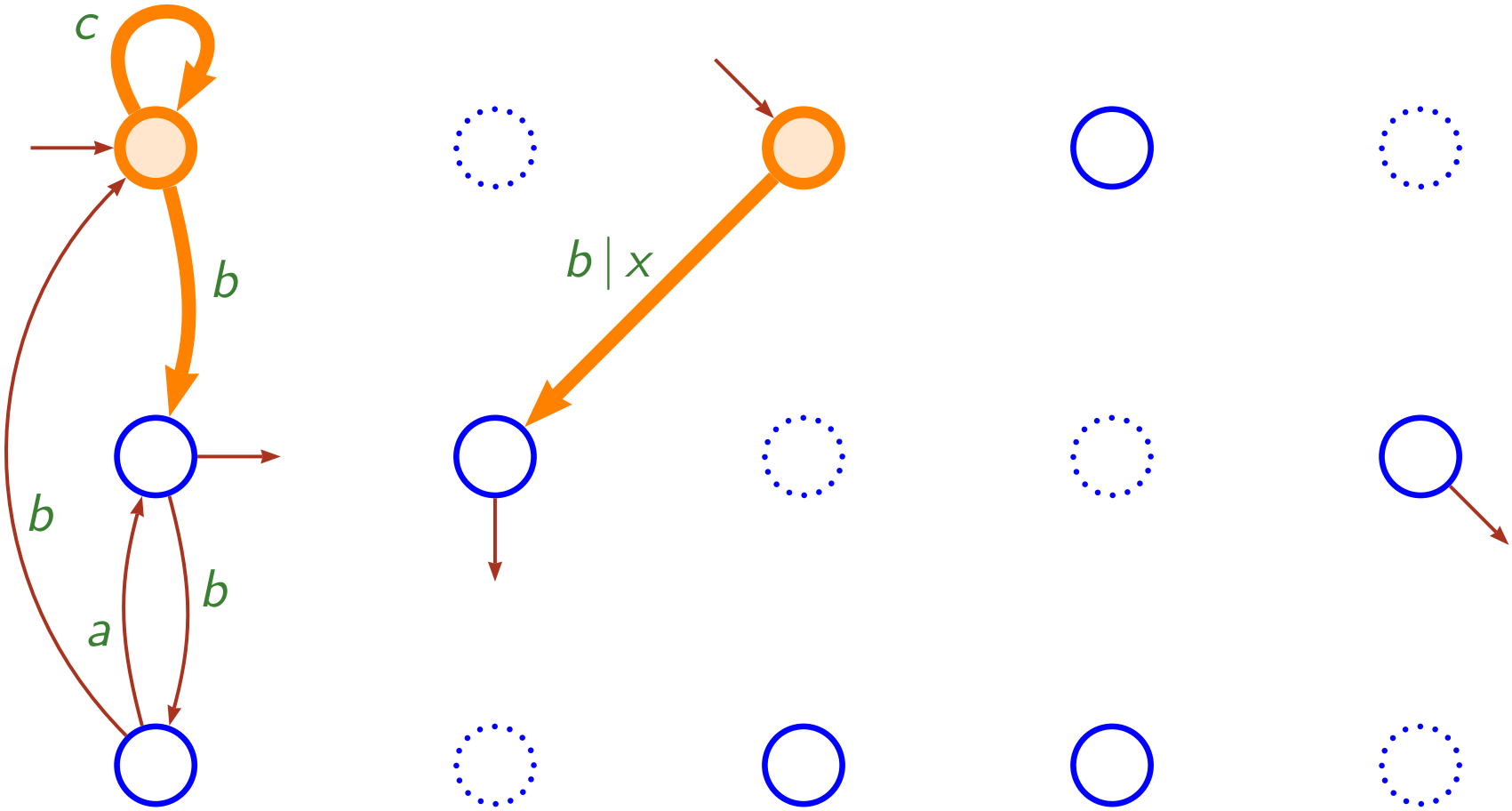

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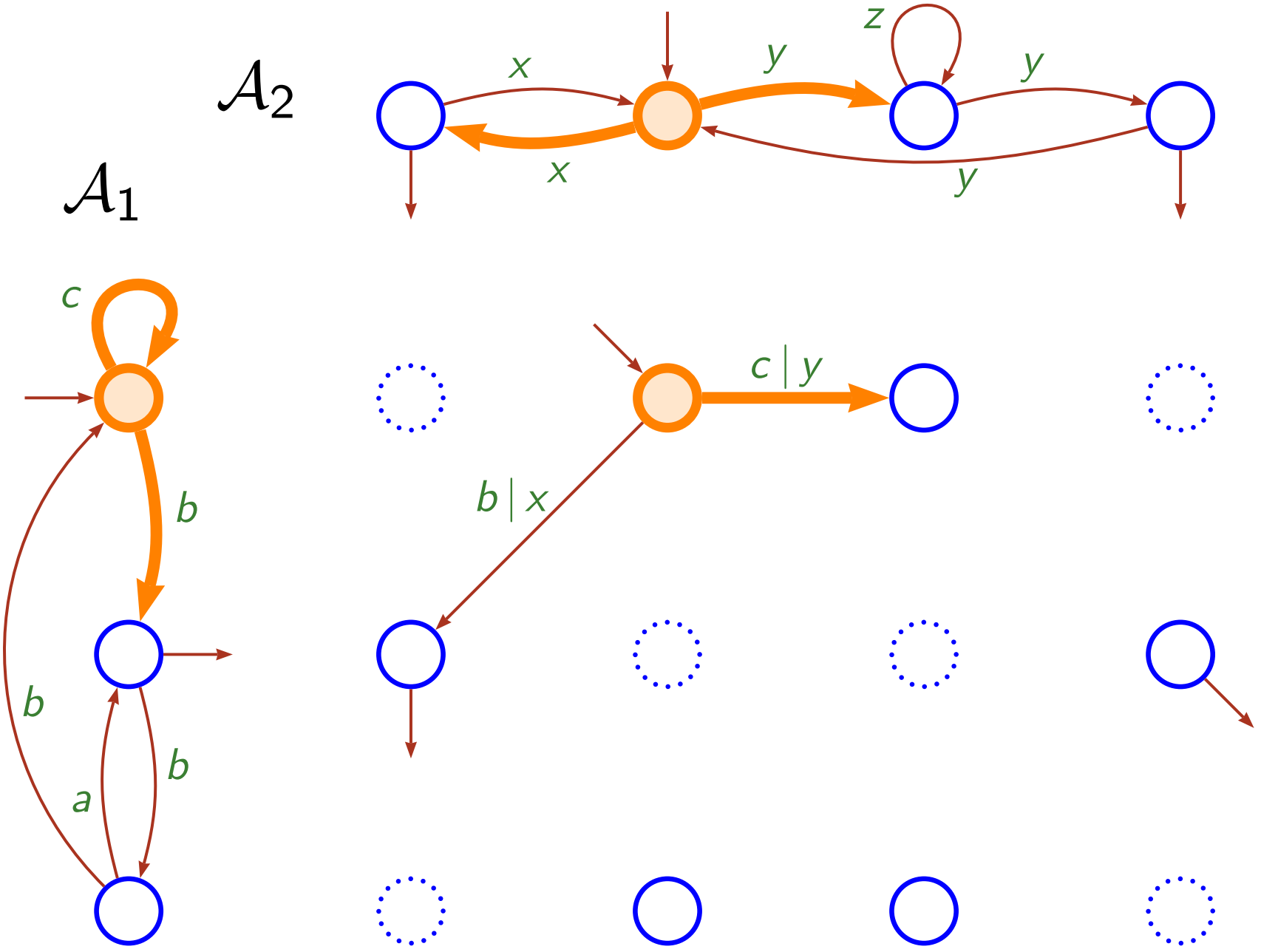

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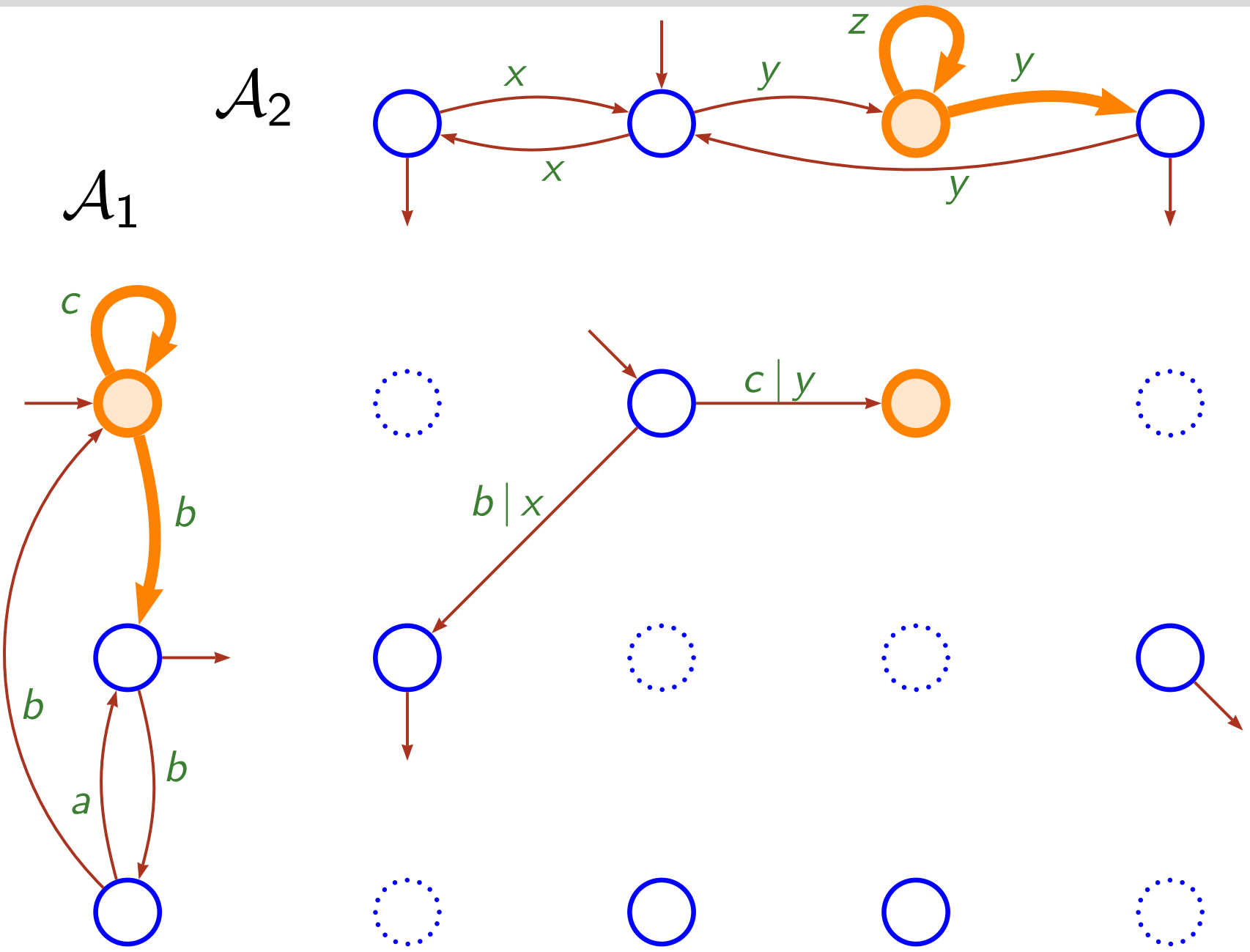

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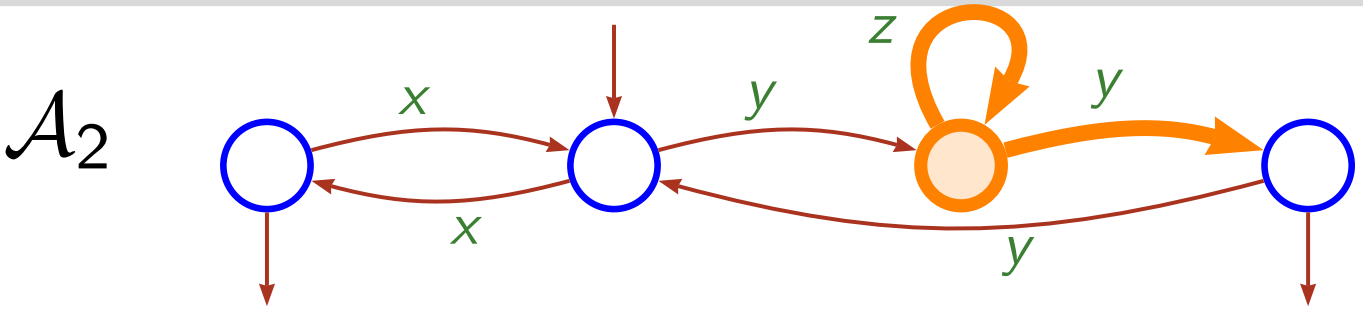

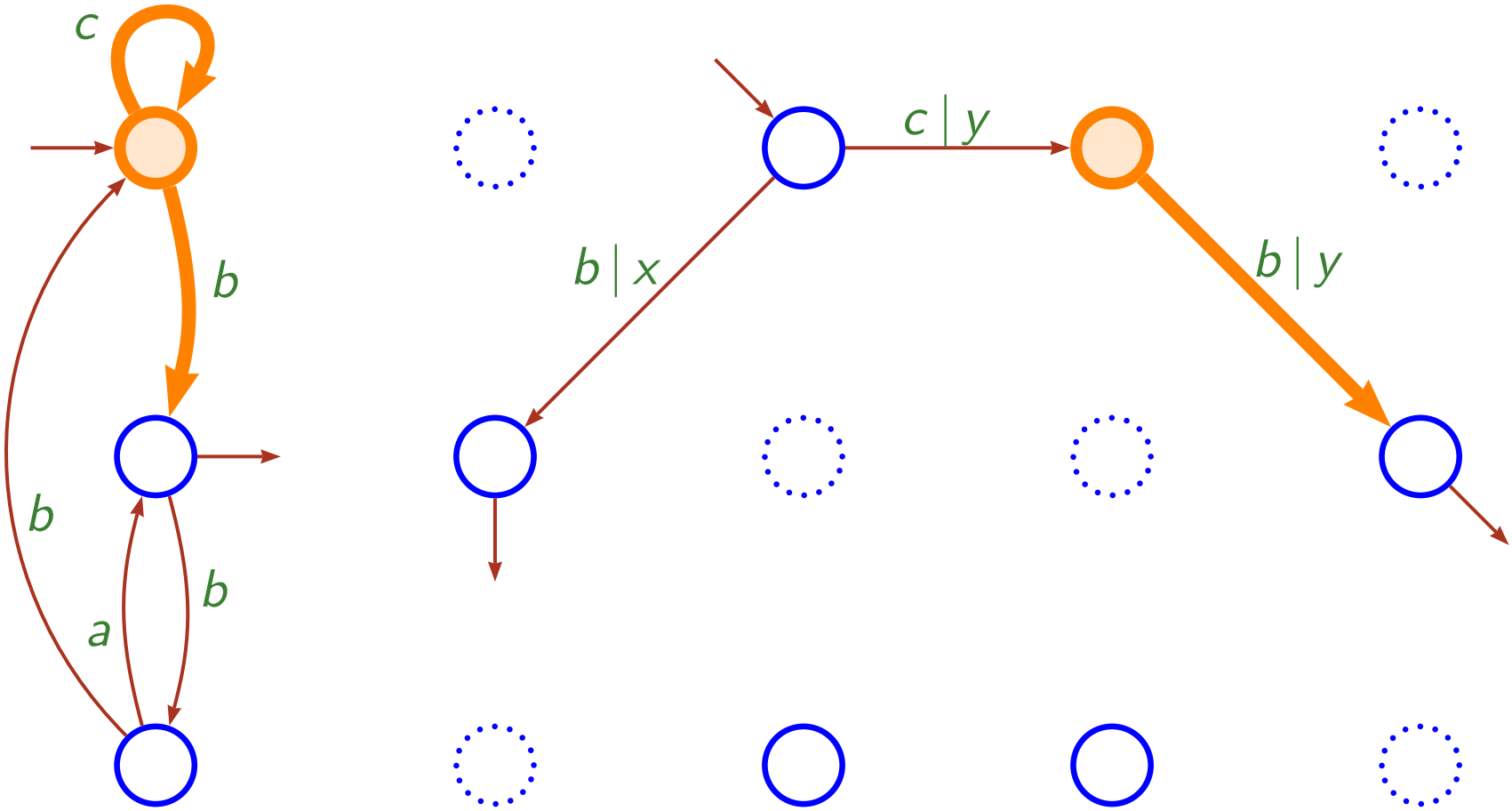

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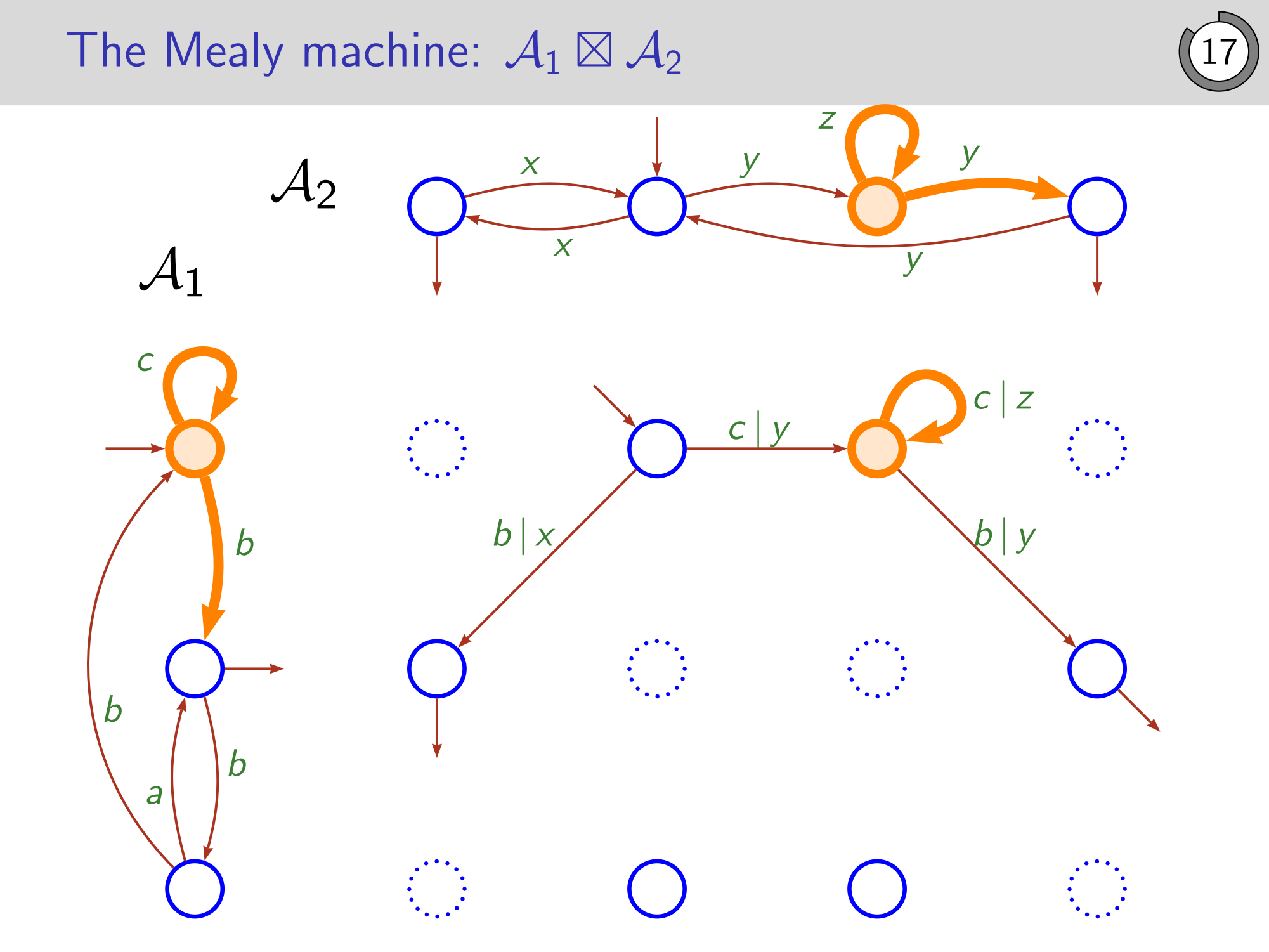

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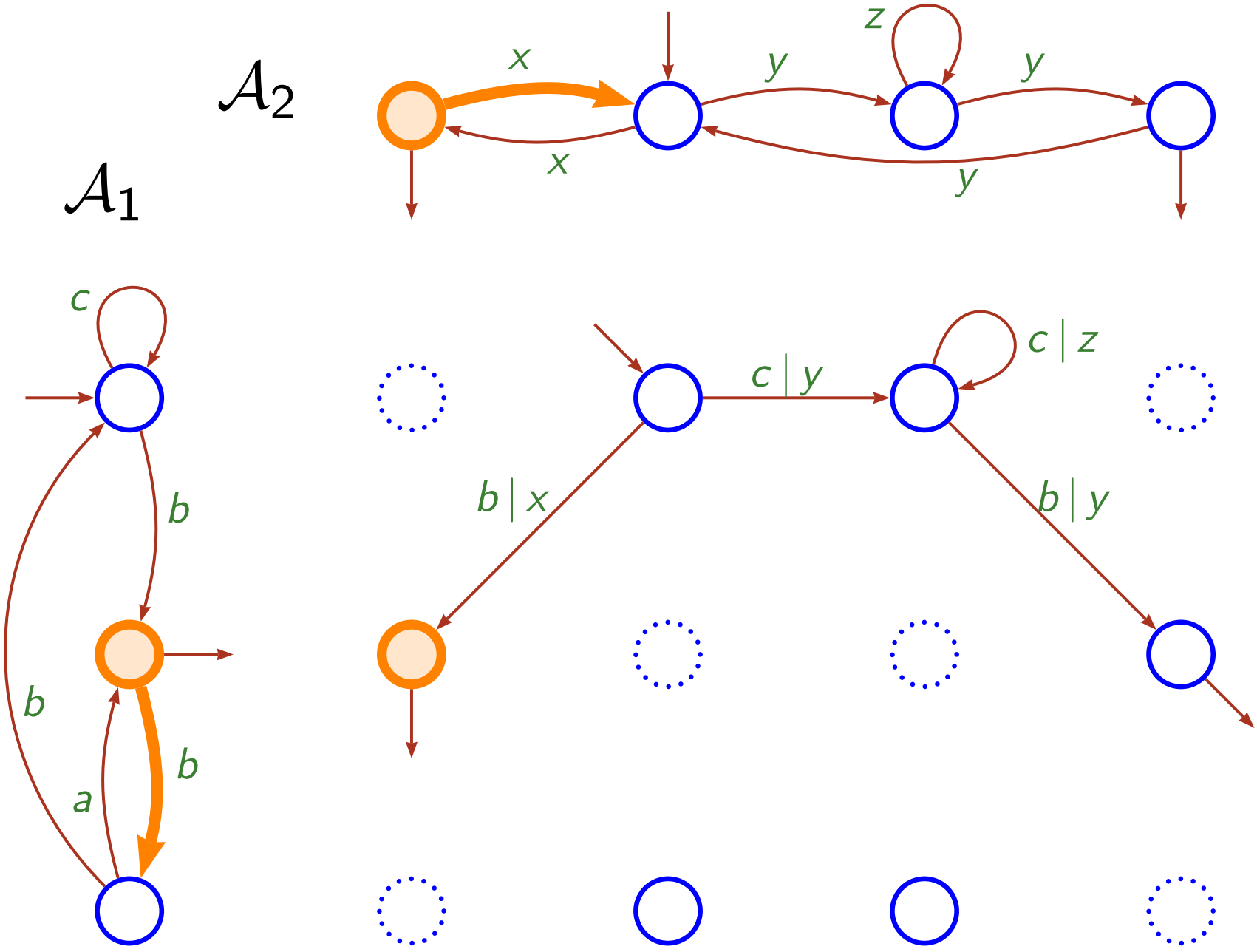

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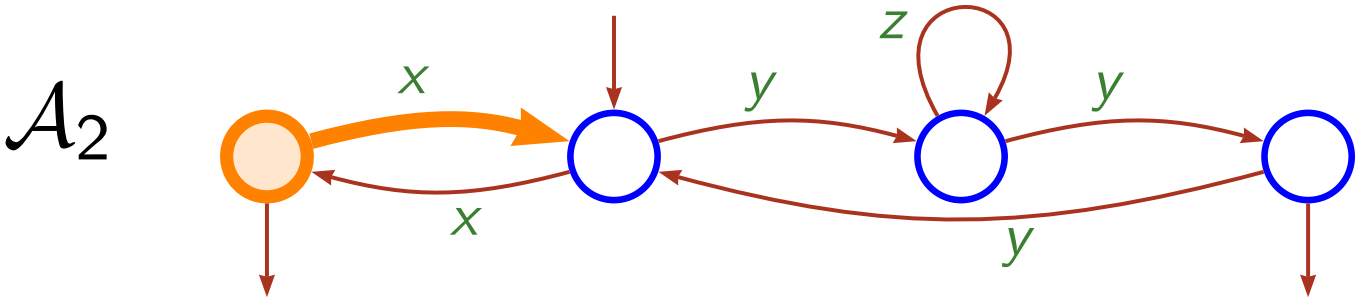

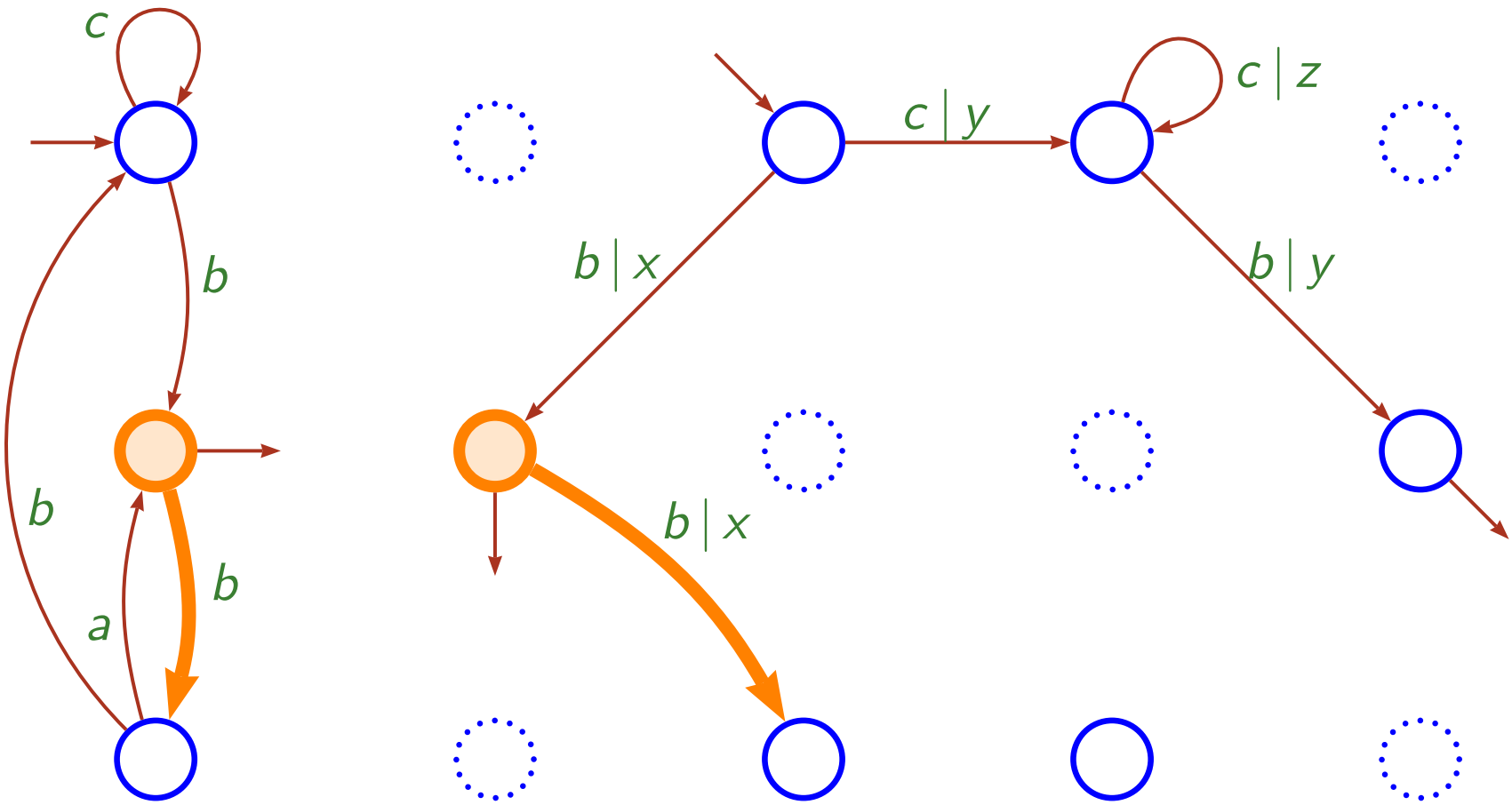

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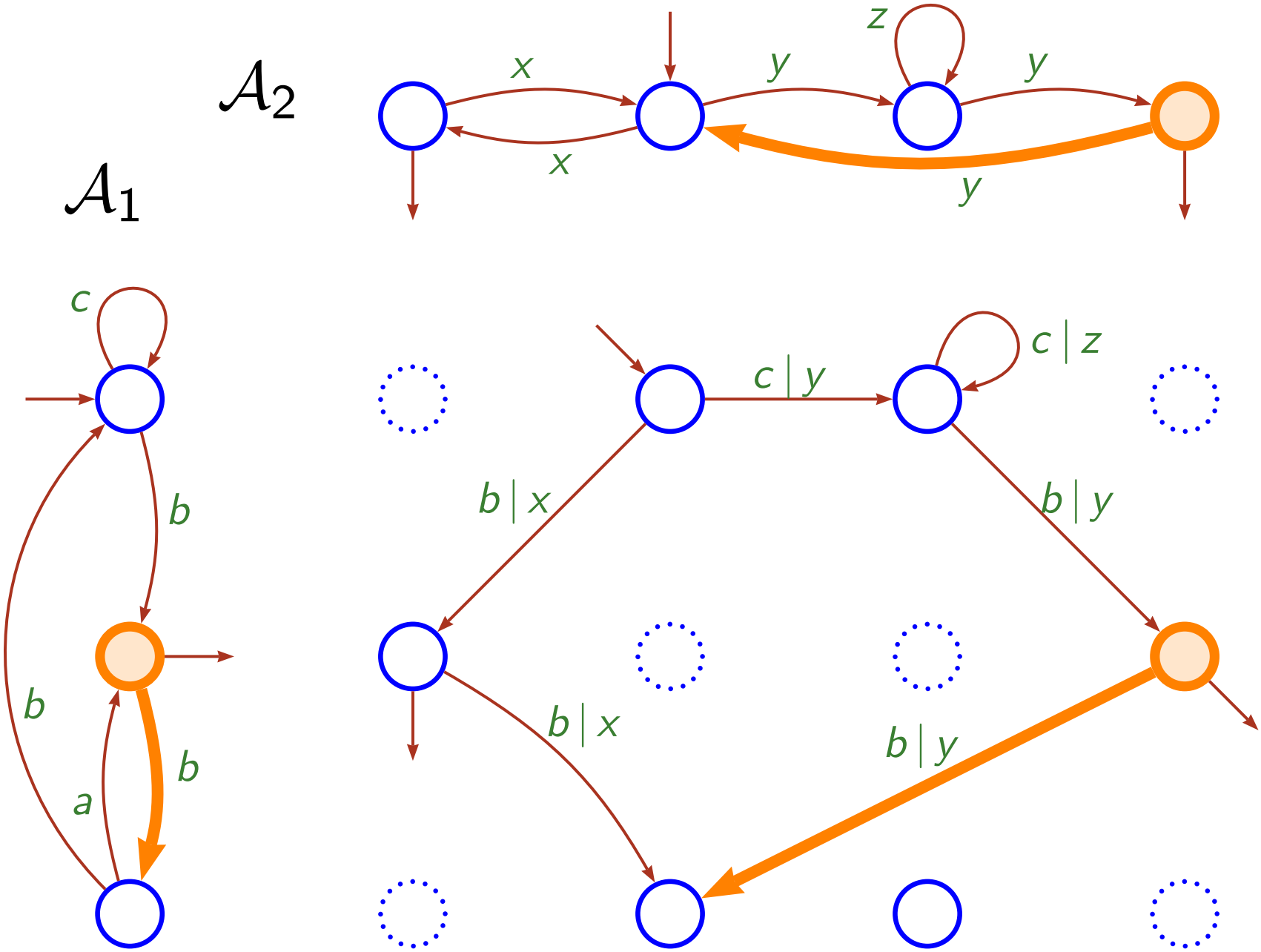

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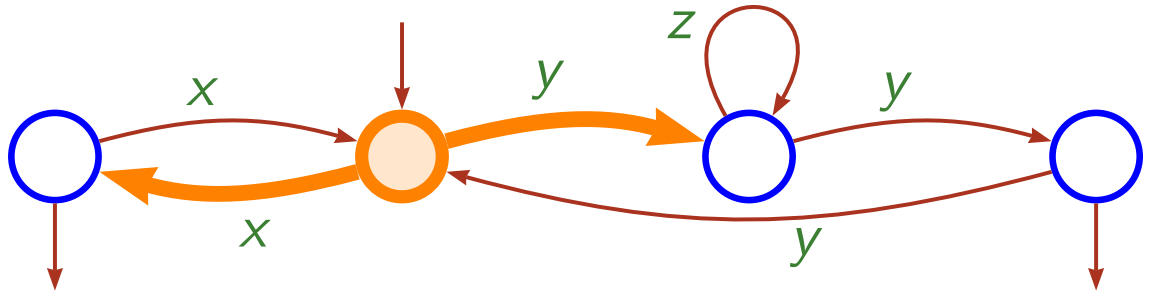
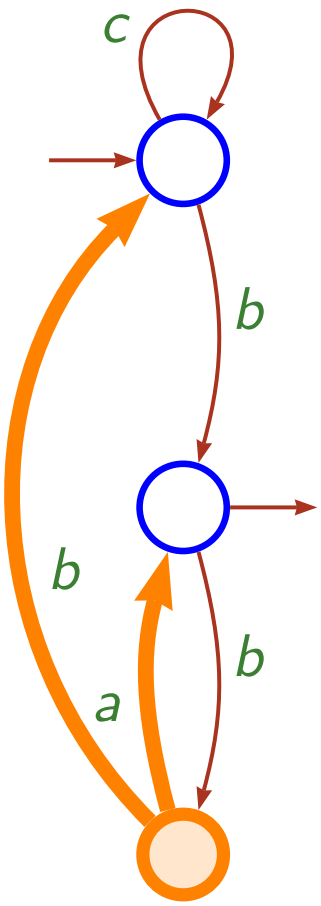
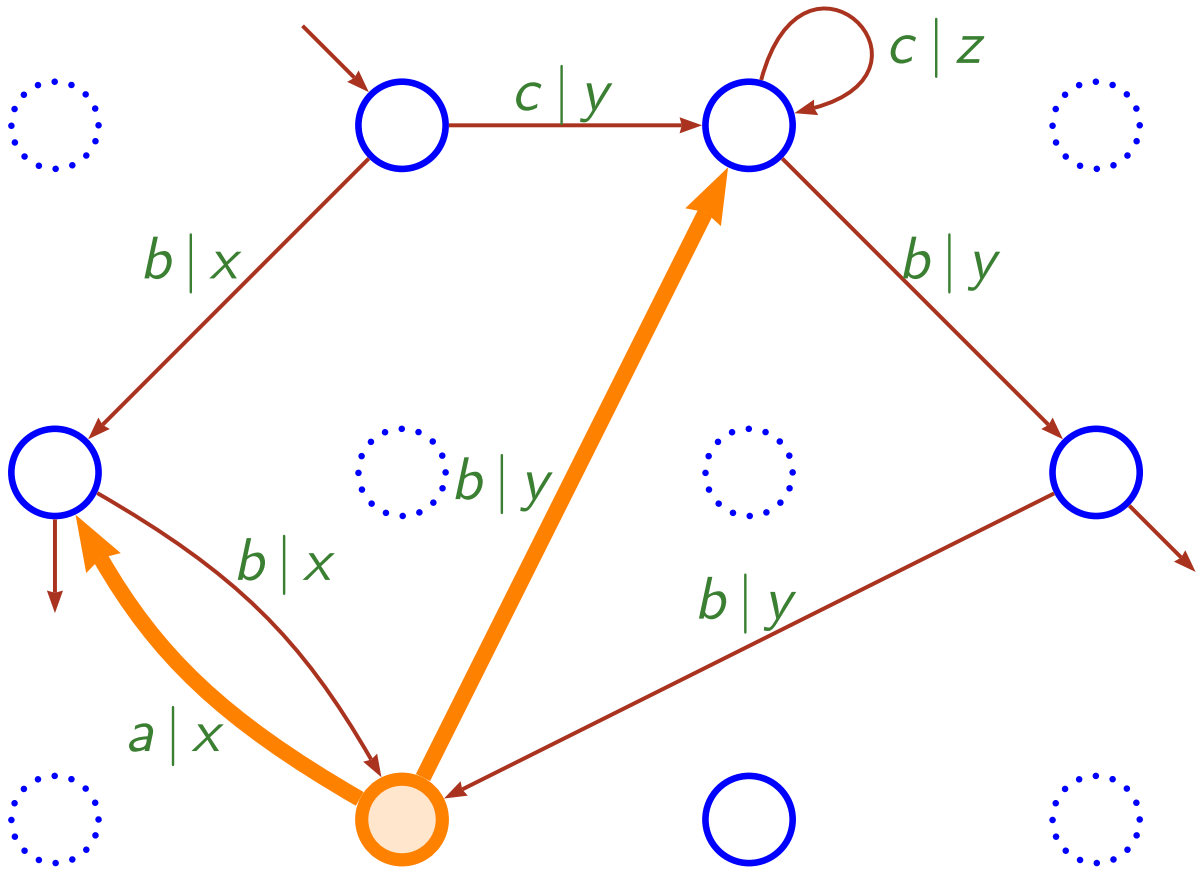
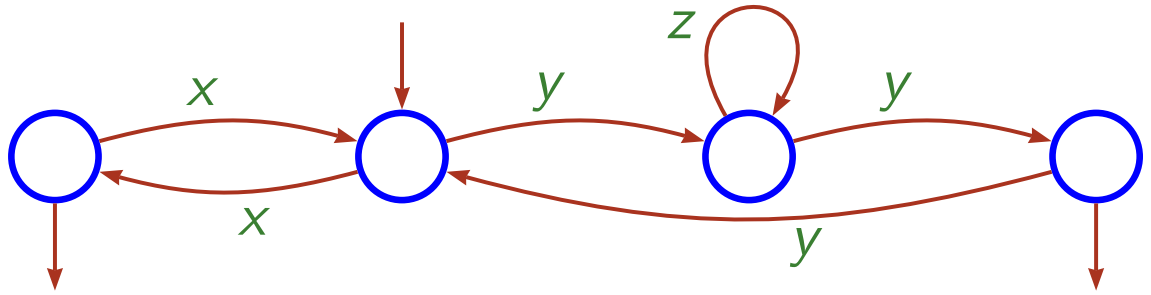
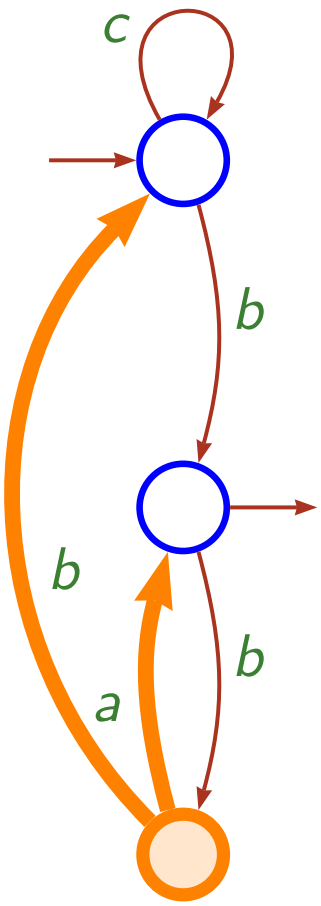


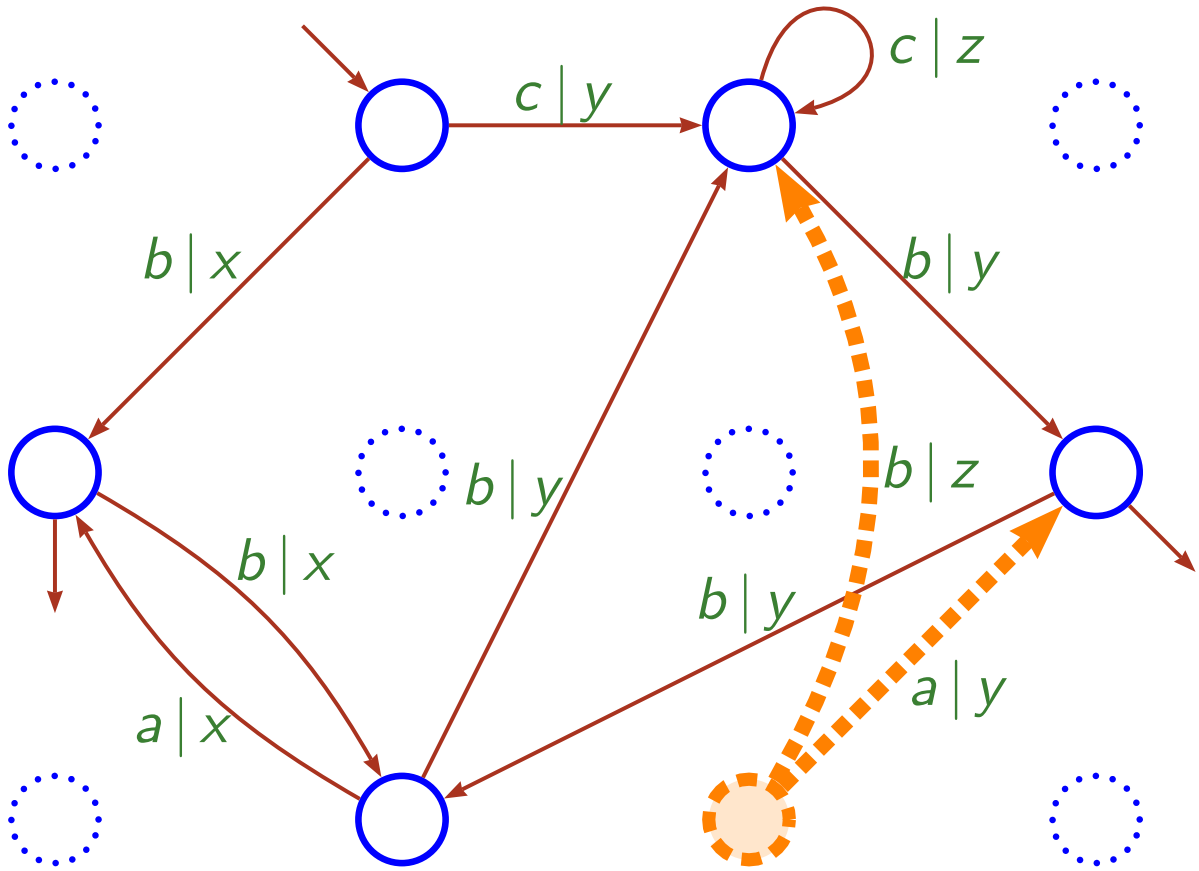




The Mealy machine: $\mathcal{A}_1 \boxtimes \mathcal{A}_2$







Two words on U-systems

Basis: $U = u_0, u_1, u_2, \ldots, u_i, \ldots$

Evaluation function: $\pi(a_k \cdots a_2 a_1 a_0) = \sum_{i=0}^{k} a_i \, u_i$

Representation of n: $\langle n \rangle_U$ is computed by the "greedy algorithm"

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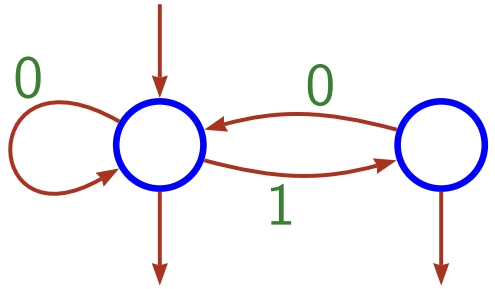
Example: Fibonacci Numeration System

$F = 1, 2, 3, 5, 8, 13, 21, \cdots$

$F_{(n+2)} = F_{(n+1)} + F_n$

$\langle 30 \rangle_F = 1\,0\,1\,0\,0\,0\,1 \quad (30 = 21 + 8 + 1 = F_6 + F_4 + F_0)$



Automaton accepting $0^* L(F)$

Theorem

U: a U-system.
$L(U) = \{\langle n \rangle \mid n \in \mathbb{N}\}$.
$\mathcal{A}$: the minimal DFA accepting $0^* L(U)$.

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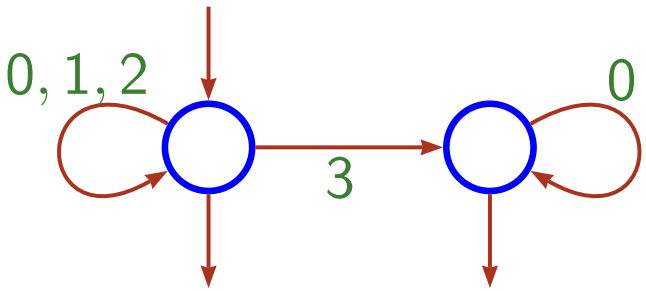
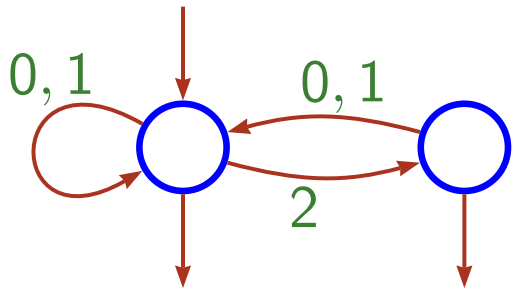


$H = 1,\ 4,\ 13,\ 40,\ 121\ \cdots$

$H_{(n+2)} = 4\,H_{(n+1)} - 3\,H_n$



$D = 1,\ 3,\ 8,\ 22,\ 60\ \cdots$

$D_{(n+2)} = 2\,D_{(n+1)} + 2\,D_n$

Conclusion

Surminimal vs Minimal automaton

- Fewer states
- Fewer letters
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Numeration Systems